Review Problems for Midterm 2

Consider the initial value problem

$$t^2y'' + 5ty' + 4y = \ln t, \quad y(1) = 2, \quad y'(1) = 3.$$

Solve this problem three different ways:

1. Observe that t^r is a solution of the homogeneous equation for some choice of r. Then use variation of parameters or reduction of order to get a first order nonhomogeneous equation, which you solve by first order techniques. Write down the general solution to the original nonhomogenous equation and plug in the initial conditions to find the solution.

2. The solution is given by a power series expansion about the point $x_0 = 1$. Find the recursion relation on the coefficients, use the initial conditions to get the first two coefficients then find the next three coefficients.

3. change variables using $x = \ln t$ or $t = e^x$. This gives a second order equation, where the homogeneous part has constant coefficients. Solve the homogenous equation without using series. Use the method of undetermined coefficients to find a particular solution. Find the general solution to the homogeneous equation and then the general solution to the original equation. Finally plug in the initial conditions to find the desired solution.

4. Check each of the three answers above by differentiating and by computing $y(1)$ and $y'(1)$.

5. We have three existence and uniqueness theorems: one treats a general second order equation with a Lipschitz condition in a rectangle, one treats second order with a Lipschitz condition in a strip and one treats linear second order with analytic coefficients. Discuss how each of these theorems apply to the equation above: where does this differential equation satisfy the hypotheses and what do the theorems say about the solutions? For the first two theorems, give formulae for the Picard iteration explicitly: how to obtain the functions at the $(n+1)^{st}$ step from the n^{th} step.

6. In the course of solving the differential equation above, you found two different solutions of the homogenous equation (hopefully they were the same for the different techiques!). Find the Wronskian of these two solutions. Is the Wronskian ever zero?

7. Prove that the Wronskian of two solutions of a second order linear homogeneous equation is either identically zero or never zero on the interval where a solution is guaranteed to exist. Hint: wtite

down the fact that y_1 and y_2 satisfy the homogeneous equation, take a combination of these two equations that eliminates the terms with no derivatives in them and obtain a first order differential equation for the Wronskian.

In some sense we were lucky that there was an easily observed solution for the first technique and that the equation became much simpler for the third technique.

8. Problem 11 page 217, section 3.8. If you use a formula in the text to do part (b), then prove the formula is correct.