## Analytic Functions

## $\S 2$. Power Series.

This note is about complex power series. Here is the primary example:

$$
\sum_{n=0}^{\infty} z^{n}
$$

This series is important to understand because its behavior is typical of all power series (defined shortly) and because it is one of the few series we can actually add up explicitly. The partial sums

$$
S_{m}=\sum_{n=0}^{m} z^{n}=1+z+z^{2}+\ldots+z^{m}
$$

satisfy

$$
(1-z) S_{m}=1-z^{m+1}
$$

as can be seen by multiplying out the left side and canceling. If $z \neq 1$ then

$$
S_{m}=\frac{1-z^{m+1}}{1-z}
$$

Notice that if $|z|<1$, then $\left|z^{m}\right|=|z|^{m} \rightarrow 0$ as $m \rightarrow \infty$ and so $S_{m}(z) \rightarrow 1 /(1-z)$. If $|z|>1$, then $\left|z^{m}\right|=|z|^{m} \rightarrow \infty$ and so the sum diverges for these $z$. If $|z|=1$ but $z \neq 1$ then $z^{n}$ does not tend to 0 , so the series diverges. Finally if $z=1$ then the partial sums satisfy $S_{m}=m \rightarrow \infty$, so we conclude that if $|z|<1$ then

$$
\begin{equation*}
\sum_{n=0}^{\infty} z^{n}=\frac{1}{1-z} \tag{2.1}
\end{equation*}
$$

and if $|z| \geq 1$, then the series diverges. It is important to note that the left and right sides of (2.1) are different objects. They agree in $|z|<1$, the right side is defined for all $z \neq 1$, but the left side is defined only for $|z|<1$.

The formal power series

$$
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}=a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\ldots
$$

is called a convergent power series centered (or based) at $z_{0}$ if there is an $r>0$ so that the series converges for all $z$ such that $\left|z-z_{0}\right|<r$. Note: If we plug $z=z_{0}$ into the formal power series, then we always get $a_{0}=f\left(z_{0}\right)$ (more formally, the definition of the summation notation includes the convention that the $n=0$ term equals $a_{0}$, so that we are not raising 0 to the power 0 .) The requirement for a power series to converge is stronger than convergence at just the one point $z_{0}$.

A variant of the primary example is:

$$
\frac{1}{z-a}=\frac{1}{z-z_{0}-\left(a-z_{0}\right)}=\frac{1}{-\left(a-z_{0}\right)\left(1-\left(\frac{z-z_{0}}{a-z_{0}}\right)\right)}
$$

Substituting

$$
w=\frac{z-z_{0}}{a-z_{0}}
$$

into (2.1) we obtain, when $|w|=\left|\left(z-z_{0}\right) /\left(a-z_{0}\right)\right|<1$,

$$
\begin{equation*}
\frac{1}{z-a}=\sum_{n=0}^{\infty} \frac{-1}{\left(a-z_{0}\right)^{n+1}}\left(z-z_{0}\right)^{n} \tag{2.2}
\end{equation*}
$$

In other words by (2.1), this series converges if $\left|z-z_{0}\right|<\left|a-z_{0}\right|$ and diverges if $\left|z-z_{0}\right| \geq$ $\left|a-z_{0}\right|$. The region of convergence is a disk and it is the largest disk centered at $z_{0}$ and contained in the domain of definition of $1 /(z-a)$. In particular, this function has a power series expansion based at every $z_{0} \neq a$, but different series for different base points $z_{0}$.

Definition. Suppose $\left\{f_{n}\right\}$ is a sequence of functions defined on a set $E$. Suppose $f$ is a function defined on $E$ with the property that for every $\varepsilon>0$ there exists an $N$ so that if $n \geq N$ then $\left|f_{n}(z)-f(z)\right|<\varepsilon$ for all $z \in E$. In this case we say that $f_{n}$ converges uniformly to $f$.

If $f_{n}$ are continuous functions on a set $E$ and if $f_{n}$ converges uniformly to $f$ on $E$, then $f$ is continuous on $E$. Indeed, given $\varepsilon>0$ choose $N$ as in the definition of uniformly
continuous. Fix $w \in E$. Because $f_{N}$ is continuous at $w$, we can choose $\delta>0$ so that if $|z-w|<\delta$ with $z \in E$ then $\left|f_{N}(z)-f_{N}(w)\right|<\varepsilon$. Then for $|z-w|<\delta$,

$$
|f(z)-f(w)| \leq\left|f(z)-f_{N}(z)\right|+\left|f_{N}(z)-f_{N}(w)\right|+\left|f_{n}(w)-f(w)\right|<3 \varepsilon
$$

This proves that $f$ is continuous at each $w \in E$.

Theorem 2.1 (Weierstrass M-Test). If $\left|a_{n}\left(z-z_{0}\right)^{n}\right| \leq M_{n}$ for $\left|z-z_{0}\right| \leq r$ and if $\sum M_{n}<\infty$ then $\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}$ converges uniformly and absolutely in $\left\{z:\left|z-z_{0}\right| \leq r\right\}$.

Proof. If $M>N$ then the partial sums $S_{n}(z)$ satisfy

$$
\left|S_{M}(z)-S_{N}(z)\right|=\left|\sum_{n=N+1}^{M} a_{n}\left(z-z_{0}\right)^{n}\right| \leq \sum_{n=N+1}^{M} M_{n}
$$

Since $\sum M_{n}<\infty$, we deduce $\sum_{n=N+1}^{M} M_{n} \rightarrow 0$ as $N, M \rightarrow \infty$, and so $\left\{S_{n}\right\}$ is a Cauchy sequence converging uniformly. The same proof also shows absolute convergence.

Note that the convergence depends only on the "tail" of the series so that we need only satisfy the hypotheses in the Weierstrass M-test for $n \geq n_{0}$ to obtain the conclusion.

The primary example (2.1) converges on a disk and diverges outside the disk. The next result says that disks are the only kind of region in which a power series can converge.

Theorem 2.2 (Root Test). Suppose $\sum a_{n}\left(z-z_{0}\right)^{n}$ is a formal power series. Let

$$
R=\liminf _{n \rightarrow \infty}\left|a_{n}\right|^{-\frac{1}{n}}=\frac{1}{\limsup _{n \rightarrow \infty}\left|a_{n}\right|^{\frac{1}{n}}} \in[0,+\infty]
$$

Then $\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}$
(a) converges absolutely in $\left\{z:\left|z-z_{0}\right|<R\right\}$,
(b) converges uniformly in $\left\{z:\left|z-z_{0}\right| \leq r\right\}$ for all $r<R$, and
(c) diverges in $\left\{z:\left|z-z_{0}\right|>R\right\}$.


Figure II. 3 Convergence of a power series.

The number $R$ gives a decay rate for the coefficients, in the sense that if $S<R$ then $\left|a_{n}\right| \leq S^{-n}$, for large $n$.

Proof. The idea is to compare the given series with the example (2.1), $\sum z^{n}$. If $\left|z-z_{0}\right| \leq$ $r<R$, then choose $r_{1}$ with $r<r_{1}<R$. Thus $r_{1}<\liminf \left|a_{n}\right|^{-\frac{1}{n}}$, and there is an $n_{0}<\infty$ so that $r_{1}<\left|a_{n}\right|^{-\frac{1}{n}}$ for all $n \geq n_{0}$. This implies that $\left|a_{n}\left(z-z_{0}\right)^{n}\right| \leq\left(\frac{r}{r_{1}}\right)^{n}$. But by (2.1),

$$
\sum_{n=0}^{\infty}\left(\frac{r}{r_{1}}\right)^{n}=\frac{1}{1-r / r_{1}}<\infty
$$

since $r / r_{1}<1$. Applying Weierstrass's M-test to the tail of the series ( $n \geq n_{0}$ ) proves (b). This same proof also shows absolute convergence (a) for each $z$ with $\left|z-z_{0}\right|<R$. If $\left|z-z_{0}\right|>R$, fix $z$ and choose $r$ so that $R<r<\left|z-z_{0}\right|$. Then $\left|a_{n}\right|^{-\frac{1}{n}}<r$ for infinitely many $n$ and hence

$$
\left|a_{n}\left(z-z_{0}\right)^{n}\right|>\left(\frac{\left|z-z_{0}\right|}{r}\right)^{n}
$$

for infinitely many $n$. Since $\left(\left|z-z_{0}\right| / r\right)^{n} \rightarrow \infty$ as $n \rightarrow \infty$, (c) holds.

The proof of the Root Test also shows that if the terms $a_{n}\left(z-z_{0}\right)^{n}$ of the formal power series are bounded when $z=z_{1}$ then the series converges on $\left\{z:\left|z-z_{0}\right|<\left|z_{1}-z_{0}\right|\right\}$.

The Root Test does not give any information about convergence on the circle of radius $R$. The series can converge at none, some, or all points of $\left\{z:\left|z-z_{0}\right|=R\right\}$, as the following examples illustrate.

## Examples.

(i) $\sum_{n=1}^{\infty} \frac{z^{n}}{n}$
(ii) $\sum_{n=1}^{\infty} \frac{z^{n}}{n^{2}}$
(iii) $\sum_{n=1}^{\infty} n z^{n}$
(iv) $\sum_{n=1}^{\infty} 2^{n^{2}} z^{n}$
(v) $\sum_{n=1}^{\infty} 2^{-n^{2}} z^{n}$

The reader should verify the following facts about these examples. The radius of convergence of each of the first three series is $R=1$. When $z=1$, the first series is the harmonic series which diverges, and when $z=-1$ the first series is an alternating series whose terms decrease in absolute value and hence converges. The second series converges uniformly and absolutely on $\{|z|=1\}$. The third series diverges at all points of $\{|z|=1\}$. The fourth series has radius of convergence $R=0$ and hence is not a convergent power series. The fifth example has radius of convergence $R=\infty$ and hence converges for all $z \in \mathbb{C}$.

What is the radius of convergence of the series $\sum a_{n} z^{n}$ where

$$
a_{n}= \begin{cases}3^{-n} & \text { if } n \text { is even } \\ 4^{n} & \text { if } n \text { is odd } ?\end{cases}
$$

This is an example where ratios of successive terms in the series does not provide sufficient information to determine convergence.

## §3. Analytic Functions

Definition 3.1. A function $f$ is analytic at $z_{0}$ if $f$ has a power series expansion valid in a neighborhood of $z_{0}$. This means that there is an $r>0$ and a power series $\sum a_{n}\left(z-z_{0}\right)^{n}$ which converges in $B=\left\{z:\left|z-z_{0}\right|<r\right\}$ and satisfies

$$
f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}
$$

for all $z \in B$. A function $f$ is analytic on a set $\Omega$ if $f$ is analytic at each $z_{0} \in \Omega$.
Note that we do not require one series for $f$ to converge in all of $\Omega$. The example (2.2), $(z-a)^{-1}$, is analytic on $\mathbb{C} \backslash\{a\}$ and is not given by one series. Note that if $f$ is analytic on $\Omega$ then $f$ is continuous in $\Omega$. Indeed, continuity is a local property. To check continuity near $z_{0}$, use the series based at $z_{0}$. Since the partial sums are continuous and converge uniformly on a closed disk centered at $z_{0}$, the limit function $f$ is continuous on that disk.

A natural question at this point is: where is a power series analytic?

Theorem 3.2. If $f(z)=\sum a_{n}\left(z-z_{0}\right)^{n}$ converges on $\left\{z:\left|z-z_{0}\right|<r\right\}$ then $f$ is analytic on $\left\{z:\left|z-z_{0}\right|<r\right\}$.

Proof. Fix $z_{1}$ with $\left|z_{1}-z_{0}\right|<r$. We need to prove that $f$ has a power series expansion based at $z_{1}$. By the binomial theorem

$$
\left(z-z_{0}\right)^{n}=\left(z-z_{1}+z_{1}-z_{0}\right)^{n}=\sum_{k=0}^{n}\binom{n}{k}\left(z_{1}-z_{0}\right)^{n-k}\left(z-z_{1}\right)^{k}
$$

Hence

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a_{n}\binom{n}{k}\left(z_{1}-z_{0}\right)^{n-k}\left(z-z_{1}\right)^{k}\right] \tag{3.1}
\end{equation*}
$$

Suppose for the moment, that we can interchange the order of summation, then

$$
\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty} a_{n}\binom{n}{k}\left(z_{1}-z_{0}\right)^{n-k}\right]\left(z-z_{1}\right)^{k}
$$

should be the power series expansion for $f$ based at $z_{1}$. To justify this interchange of summation, it suffices to prove absolute convergence of (3.1). By the root test

$$
\sum_{n=0}^{\infty}\left|a_{n}\right|\left|w-z_{0}\right|^{n}
$$

converges if $\left|w-z_{0}\right|<r$. Set

$$
w=\left|z-z_{1}\right|+\left|z_{1}-z_{0}\right|+z_{0}
$$

Then $\left|w-z_{0}\right|=\left|z-z_{1}\right|+\left|z_{1}-z_{0}\right|<r$ provided $\left|z-z_{1}\right|<r-\left|z_{1}-z_{0}\right|$.


Figure II. 4 Proof of Theorem 3.2.

Thus if $\left|z-z_{1}\right|<r-\left|z_{1}-z_{0}\right|$,

$$
\begin{aligned}
\infty & >\sum_{n=0}^{\infty}\left|a_{n}\right|\left|w-z_{0}\right|^{n} \\
& =\sum_{n=0}^{\infty}\left|a_{n}\right|\left(\left|z-z_{1}\right|+\left|z_{1}-z_{0}\right|\right)^{n} \\
& =\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\left|a_{n}\right|\binom{n}{k}\left|z_{1}-z_{0}\right|^{n-k}\left|z-z_{1}\right|^{k}\right]
\end{aligned}
$$

as desired.

Another natural question is: Can an analytic function have more than one power series expansion based at $z_{0}$ ?

Theorem 3.3 (Uniqueness of Series). Suppose

$$
\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}=\sum_{n=0}^{\infty} b_{n}\left(z-z_{0}\right)^{n}
$$

for all $z$ such that $\left|z-z_{0}\right|<r$ where $r>0$. Then $a_{n}=b_{n}$ for all $n$.

Proof. Set $c_{n}=a_{n}-b_{n}$. The hypothesis implies that $\sum_{n=0}^{\infty} c_{n}\left(z-z_{0}\right)^{n}=0$ and we need to show that $c_{n}=0$ for all $n$. Suppose $c_{m}$ is the first non-zero coefficient. Set

$$
F(z) \equiv \sum_{n=0}^{\infty} c_{n+m}\left(z-z_{0}\right)^{n}=\left(z-z_{0}\right)^{-m} \sum_{n=m}^{\infty} c_{n}\left(z-z_{0}\right)^{n} .
$$

The series for $F$ converges in $0<\left|z-z_{0}\right|<r$ because we can multiply the terms of the series on the right side by the non-zero number $\left(z-z_{0}\right)^{-m}$ and not affect convergence. By the root test, the series for $F$ converges in a disk and hence in $\left\{\left|z-z_{0}\right|<r\right\}$. Since $F$ is continuous and $c_{m} \neq 0$, there is a $\delta>0$ so that if $\left|z-z_{0}\right|<\delta$, then

$$
\left|F(z)-F\left(z_{0}\right)\right|=\left|F(z)-c_{m}\right|<\left|c_{m}\right| / 2
$$

If $F(z)=0$, then we obtain the contradiction $\left|-c_{m}\right|<\left|c_{m}\right| / 2$. Thus $F(z) \neq 0$ when $\left|z-z_{0}\right|<\delta$. But $\left(z-z_{0}\right)^{m}=0$ only when $z=z_{0}$, and thus

$$
\sum_{n=0}^{\infty} c_{n}\left(z-z_{0}\right)^{n}=\left(z-z_{0}\right)^{m} F(z) \neq 0
$$

when $0<\left|z-z_{0}\right|<\delta$, contradicting our assumption on $\sum c_{n}\left(z-z_{0}\right)^{n}$.

Notice that the proof of Theorem 3.3 shows that if $f$ is analytic at $z_{0}$ then for some $\delta>0$, either $f(z) \neq 0$ when $0<\left|z-z_{0}\right|<\delta$ or $f(z)=0$ for all $z$ such that $\left|z-z_{0}\right|<\delta$. This is because $f$ behaves like the first non-zero term in its power series based at $z_{0}$, for $\left|z-z_{0}\right|$ sufficiently small. If $f(a)=0$, then $a$ is called a zero of $f$.

There are plenty of continuous functions for which the zeros are not isolated in this manner. For example $x \sin (1 / x)$.

## $\S 4$. Elementary Operations with Analytic Functions

Theorem 4.1. If $f$ and $g$ are analytic at $z_{0}$ then so are

$$
f+g, \quad f-g, \quad c f(\text { where } c \text { is a constant), and } \quad f g .
$$

If $h$ is analytic at $f\left(z_{0}\right)$ then $(h \circ f)(z) \equiv h(f(z))$ is analytic at $z_{0}$.

Proof. The first three follow from the fact that the partial sums are absolutely convergent near $z_{0}$, together with the associative, commutative and distributive laws applied to the partial sums. Here we have used the fact that absolutely convergent complex series can be rearranged, which follows from the same statement for real series by considering real and imaginary parts. To prove that the product of two analytic functions is analytic, multiply $f(z)=\sum a_{n}\left(z-z_{0}\right)^{n}$ and $g=\sum b_{n}\left(z-z_{0}\right)^{n}$ as if they were polynomials to obtain:

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n} \sum_{k=0}^{\infty} b_{k}\left(z-z_{0}\right)^{k}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{k} b_{n-k}\right)\left(z-z_{0}\right)^{n} \tag{4.1}
\end{equation*}
$$

which is called the Cauchy product of the two series. Why is this formal computation valid? If the series for $f$ and the series for $g$ converge absolutely then because we can rearrange non-negative convergent series

$$
\infty>\sum_{n=0}^{\infty}\left|a_{n}\right|\left|z-z_{0}\right|^{n} \sum_{k=0}^{\infty}\left|b_{k}\right|\left|z-z_{0}\right|^{k}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left|a_{k}\right|\left|b_{n-k}\right|\right)\left|z-z_{0}\right|^{n}
$$

This says that the series on the right-hand side of (4.1) is absolutely convergent and therefore can be arranged to give the left-hand side of (4.1). To put it another way, the doubly indexed sequence $a_{n} b_{k}\left(z-z_{0}\right)^{n+k}$ can be added up two ways: If we add along diagonals: $n+k=m$, for $m=0,1,2, \ldots$, we obtain the partial sums of the right-hand side of (4.1). If we add along partial rows and columns $n=m, k=0, \ldots, m$, and $k=m$, $n=0, \ldots, m-1$, for $m=1,2, \ldots$, we obtain the product of the partial sums for the series on the left-hand side of (4.1). Since the series is absolutely convergent (as can be seen by using the latter method of summing the doubly indexed sequence of absolute values), the limits are the same.

To prove that you can compose analytic functions where it makes sense, suppose $f(z)=\sum a_{n}\left(z-z_{0}\right)^{n}$ is analytic at $z_{0}$ and suppose $h(z)=\sum b_{n}\left(z-a_{0}\right)^{n}$ is analytic at $a_{0}=f\left(z_{0}\right)$. The sum

$$
\begin{equation*}
\sum_{m=1}^{\infty}\left|a_{m}\right|\left|z-z_{0}\right|^{m-1} \tag{4.2}
\end{equation*}
$$

converges in $\left\{z: 0<\left|z-z_{0}\right|<r\right\}$ for some $r>0$ since the series for $f$ is absolutely convergent, and $\left|z-z_{0}\right|$ is non-zero. By the root test (set $k=m-1$ ), this implies that the series (4.2) converges uniformly in $\left\{\left|z-z_{0}\right| \leq r_{1}\right\}$, for $r_{1}<r$, and hence is bounded in $\left\{\left|z-z_{0}\right| \leq r_{1}\right\}$. Thus there is a constant $M<\infty$ so that

$$
\sum_{m=1}^{\infty}\left|a_{m}\right|\left|z-z_{0}\right|^{m} \leq M\left|z-z_{0}\right|
$$

if $\left|z-z_{0}\right|<r_{1}$, and so

$$
\sum_{m=0}^{\infty}\left|b_{m}\right|\left(\sum_{n=1}^{\infty}\left|a_{n}\right|\left|z-z_{0}\right|^{n}\right)^{m} \leq \sum_{m=0}^{\infty}\left|b_{m}\right|\left(M\left|z-z_{0}\right|\right)^{m}<\infty
$$

for $\left|z-z_{0}\right|$ sufficiently small, by the absolute convergence of the series for $h$. This proves absolute convergence for the composed series, and thus we can rearrange the doublyindexed series for the composition so that it is a (convergent) power series.

As a consequence, if $f$ is analytic at $z_{0}$ and $f\left(z_{0}\right) \neq 0$ then composing with the function $1 / z$ which is analytic on $\mathbb{C} \backslash\{0\}$, we conclude $1 / f$ is analytic at $z_{0}$. A rational function
$r$ is the ratio

$$
r(z)=\frac{p(z)}{q(z)}
$$

where $p$ and $q$ are polynomials. The rational function $r$ is then analytic on $\{z: q(z) \neq 0\}$.

Definition 4.2. If $f$ is defined in a neighborhood of $z$ then

$$
f^{\prime}(z)=\lim _{w \rightarrow z} \frac{f(w)-f(z)}{w-z}
$$

is called the (complex) derivative of $f$, provided the limit exists.

The function $\bar{z}$ does not have a (complex) derivative. If $n$ is a non-negative integer,

$$
\left(z^{n}\right)^{\prime}=n z^{n-1} .
$$

The next Theorem says that you can differentiate power series term-by-term.

Theorem 4.3. If $f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}$ converges in $B=\left\{z:\left|z-z_{0}\right|<r\right\}$ then $f^{\prime}(z)$ exists for all $z \in B$ and

$$
f^{\prime}(z)=\sum_{n=1}^{\infty} n a_{n}\left(z-z_{0}\right)^{n-1}=\sum_{n=0}^{\infty}(n+1) a_{n+1}\left(z-z_{0}\right)^{n},
$$

for $z \in B$. Moreover the series for $f^{\prime}$ based at $z_{0}$ has the same radius of convergence as the series for $f$.

Proof. If $0<|h|<r$ then

$$
\frac{f\left(z_{0}+h\right)-f\left(z_{0}\right)}{h}-a_{1}=\frac{\sum_{n=0}^{\infty} a_{n} h^{n}-a_{0}}{h}-a_{1}=\sum_{n=2}^{\infty} a_{n} h^{n-1}=\sum_{n=1}^{\infty} a_{n+1} h^{n}
$$

By the root test, the region of convergence for the series $\sum a_{n+1} h^{n}$ is a disk centered at 0 and hence it converges uniformly in $\left\{h:|h| \leq r_{1}\right\}$, if $r_{1}<r$. In particular, $\sum a_{n+1} h^{n}$ is continuous at 0 and hence

$$
\lim _{h \rightarrow 0} \sum_{n=1}^{\infty} a_{n+1} h^{n}=0
$$

This proves that $f^{\prime}\left(z_{0}\right)$ exists and equals $a_{1}$.
By Theorem 3.2, $f$ has a power series expansion about each $z_{1}$ with $\left|z_{1}-z_{0}\right|<r$ given by

$$
\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty} a_{n}\binom{n}{k}\left(z_{1}-z_{0}\right)^{n-k}\right]\left(z-z_{1}\right)^{k}
$$

Therefore $f^{\prime}\left(z_{1}\right)$ exists and equals the coefficient of $z-z_{1}$

$$
f^{\prime}\left(z_{1}\right)=\sum_{n=1}^{\infty} a_{n}\binom{n}{1}\left(z_{1}-z_{0}\right)^{n-1}=\sum_{n=1}^{\infty} a_{n} n\left(z_{1}-z_{0}\right)^{n-1}
$$

By the root test and the fact that $n^{\frac{1}{n}} \rightarrow 1$, the series for $f^{\prime}$ has exactly the same radius of convergence as the series for $f$.

Since the series for $f^{\prime}$ has the same radius of convergence as the series for $f$, we obtain the following corollary.

Corollary 4.4. An analytic funtion $f$ has derivatives of all orders. Moreover if $f$ is equal to a convergent power series on $B=\left\{z:\left|z-z_{0}\right|<r\right\}$ then the power series is given by

$$
f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}\left(z_{0}\right)}{n!}\left(z-z_{0}\right)^{n}
$$

for $z \in B$.
By definition of the symbols, the $n=0$ term in the series is $f\left(z_{0}\right)$.

Proof. If $f(z)=\sum_{n=0}^{\infty} a_{n}\left(z-z_{0}\right)^{n}$, then we proved in Theorem 4.2 that $a_{1}=f^{\prime}\left(z_{0}\right)$ and

$$
f^{\prime}(z)=\sum_{n=1}^{\infty} n a_{n}\left(z-z_{0}\right)^{n-1}
$$

Applying Theorem 4.3 to $f^{\prime}(z)$, we obtain $2 a_{2}=\left(f^{\prime}\right)^{\prime}\left(z_{0}\right) \equiv f^{\prime \prime}\left(z_{0}\right)$ and by induction

$$
n!a_{n}=f^{(n)}\left(z_{0}\right)
$$

If $f$ is analytic in a region $\Omega$ with $f^{\prime}(z)=0$ for all $z \in \Omega$ then by Corollary 4.4, $f$ is constant. In fact, if $f$ is non-constant then by Theorem 4.3, $f^{\prime}$ is analytic and so the zeros
of $f^{\prime}$ must be isolated. A useful consequence is that if $f$ and $g$ are analytic with $f^{\prime}=g^{\prime}$, then $f-g$ is constant.

Corollary 4.6. If $f(z)=\sum a_{n}\left(z-z_{0}\right)^{n}$ converges in $B=\left\{z:\left|z-z_{0}\right|<r\right\}$ then the power series

$$
F(z)=\sum_{n=0}^{\infty} \frac{a_{n}}{n+1}\left(z-z_{0}\right)^{n+1}
$$

converges in $B$ and satisfies

$$
F^{\prime}(z)=f(z)
$$

for $z \in B$.

The series for $F$ has the same radius of convergence as the series for $f$, by Theorem 4.3 or by direct calculation.

## §5. Exercises

## A

1. Check that examples (i)-(v) in Section 2 are correct.
2. For what values of $z$ is

$$
\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^{n}
$$

convergent? Draw a picture of the region.
3. Prove the sum, product, quotient and chain rules for analytic functions and find the derivative of $(z-a)^{-n}$, where $n$ is a positive integer and $a \in \mathbb{C}$.
4. (a) Prove that $f$ has a power series expansion about $z_{0}$ with radius of convergence $r>0$ if and only if $g(z)=\frac{f(z)-f\left(z_{0}\right)}{z-z_{0}}$ has a power series expansion about $z_{0}$, with the same radius of convergence. (How must you define $g\left(z_{0}\right)$, in terms of the coefficients of the series for $f$ to make this a true statement?)
(b) It follows from (a) that if $f$ has a power series expansion at $z_{0}$ with radius of convergence $R$ and if $\left|z-z_{0}\right| \leq r<R$ then there is a constant $C$ so that
$\left|f(z)-f\left(z_{0}\right)\right| \leq C\left|z-z_{0}\right|$. Use the same idea to show that if $f(z)=\sum a_{n}\left(z-z_{0}\right)^{n}$ then

$$
\left|f(z)-\sum_{n=0}^{k} a_{n}\left(z-z_{0}\right)^{n}\right| \leq D_{k}\left|z-z_{0}\right|^{k+1}
$$

where $D_{k}$ is a constant and $\left|z-z_{0}\right| \leq r<R$. In other words, the rate of convergence of the series is given by $\left|z-z_{0}\right|^{k+1}$
5. Define $e^{z}=\sum_{n=0}^{\infty} \frac{z^{n}}{n!}$. Show
(a) this series converges for all $z \in C$
(b) $e^{z} e^{w}=e^{z+w}$
(c) Define $\cos \theta=\frac{1}{2}\left(e^{i \theta}+e^{-i \theta}\right)$ and $\sin \theta=\frac{1}{2 i}\left(e^{i \theta}-e^{-i \theta}\right)$, so that $e^{i \theta}=\cos \theta+i \sin \theta$. Using the series for $e^{z}$ show that you obtain the same series expansions as you learned in calculus for sin and cos. Check by multiplying out the definitions that $\cos ^{2} \theta+\sin ^{2} \theta=1$, so that $e^{i \theta}$ is a point on the unit circle corresponding to the cartesian coordinate $(\cos \theta, \sin \theta)$.
(d) $\left|e^{z}\right|=e^{\operatorname{Re} z}$ and $\arg e^{z}=\operatorname{Im} z$. If $z$ is a non-zero complex number then $z=r e^{i t}$, where $r=|z|$ and $t=\arg z$. Moreover, $z^{n}=r^{n} e^{i n t}$.
(e) $e^{z}=1$ only when $z=2 \pi k i$ for some integer $k$.
(f) $\frac{d}{d z} e^{z}=e^{z}$.
(g) $\int_{0}^{2 \pi} e^{i n t} d t=0$, if $n$ is a non-zero integer.
6. Suppose $f(z)=\sum_{n=1}^{\infty} a_{k} z^{k}$ converges in $|z|<r$. Suppose also that $|f(1 / n)|<e^{-n}$ for all $n \geq n_{0}$. Prove $f(z)=0$ for all $|z|<r$.

