1. Find the Schmidt decomposition and the singular value decomposition of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$.

   NEXT TIME GET A BETTER MATRIX (easier singular values)

2. Solve the initial value problem

$$\frac{dx_1}{dt} + 2x_1 - x_2 = 0 \qquad x_1(0) = 6$$
$$\frac{dx_2}{dt} - x_1 + 2x_2 = 0 \qquad x_2(0) = 7$$

   as follows:

   (i) Write the problem in vector form:

$$\frac{d}{dt}X + A \cdot X = O \qquad X(0) = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.$$

   (ii) Find an orthonormal basis of eigenvectors of the matrix A.

   (iii) Find two independent solutions $X = X_1(t)$ and $X = X_2(t)$ of $dX/dt + A \cdot X = O$ with

$$X_1(t) = f_1(t)B_1 \text{ and } X_2(t) = f_2(t)B_2\,,$$

   where $\{B_1, B_2\}$ is an orthonormal basis of $\mathbb{R}^2$, and f_1 and f_2 are real valued functions.

   (iv) Using (ii), write the general solution in the form

$$X(t) = B \cdot \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

   where the columns of B are B_1 and B_2.

   (v) Show that $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^t \cdot \begin{pmatrix} 6 \\ 7 \end{pmatrix}$.

   (vi) Put this all together to write the solution in the form

$$X(t) = B \cdot \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \cdot B^t \cdot X_0$$

3. Solve the initial value problem

$$\begin{array}{ll} x_1' = x_2 + x_3, & x_1(0) = 3 \\ x_2' = x_1 + x_3, & x_2(0) = 0 \\ x_3' = x_1 + x_2, & x_3(0) = 0 \end{array}$$

   by finding an orthonormal basis of eigenvectors of the appropriate matrix.

   Hint: $t^3 - 3t - 2 = (t-2)(t+1)^2$

4. Let $A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ be a non-zero 3×3 skew symmetric matrix.

   (i) Show that the characteristic polynomial of A is of the form

$$p_A(t) = t^3 + (a^2 + b^2 + c^2)t = t(t - \lambda_1)(t - \lambda_2) = t(t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2)\,,$$

   and from this conclude that the spectrum of A is of the form $\{0, \lambda_1, \lambda_2\}$, with

$$\lambda_1 + \lambda_2 = 0 \text{ and } \lambda_1\lambda_2 = a^2 + b^2 + c^2\,,.$$

   Conclude that $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, with $\omega = \sqrt{a^2 + b^2 + c^2}$

(ii) Let $B_1 = \frac{1}{\omega}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Show that $A \cdot B_1 = O$.

(iii) Let $B_2 + iB_3$ satisfy

$$A \cdot (B_2 + iB_3) = i\omega(B_2 + iB_3) \implies A \cdot B_2 = -\omega\, B_3 \text{ and } A \cdot B_3 = \omega B_2$$

Prove that $\|B_2|| = \|B_3\|$ and $\langle B_1, B_2\rangle = \langle B_1, B_3\rangle = \langle B_3, B_3\rangle = 0$. Hence, (after rescaling B_2 and B_3 if necessary) $\{B_1, B_2, B_3\}$ is an orthogonal basis for $\mathbb{R}^3$ with respect to the standard scalar product.

Hint: *Show that $X^t \cdot A \cdot X = 0$ for any vector; then show that for $i \neq j$, the scalar product $\langle B_i, B_j\rangle$ can be expressed as a multiple of X^tAX for $X = B_i$ or $X = B_j$.*

(iv) Use these facts to prove that

$$R_t = \exp(tA) = B \cdot \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega t \\ 0 & -\omega t & 0 \end{pmatrix} \cdot B^t = B \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & \sin(\omega t) \\ 0 & -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \cdot B^t\,.$$

In particular, R_t is a clockwise rotation by ωt radians about the vector B_1:

$$R_t \cdot B_1 = B_1, \quad R_t(B_2) = \cos(\omega t)B_2 - \sin(\omega t)B_3\, R_t(B_3) = \sin(\omega t)B_2 + \cos(\omega t)B_3\,.$$

5. Which of the matrices below are diagonalizable? In each case, give a one or two sentence explanation for your answer.

$$(a)\begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad (b)\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 1 & 5 & 7 \end{pmatrix} \qquad (c)\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

6. What is the shape of the set of all solutions to the equation

$$4x^2 - 2xy + 4y^2 = 1?$$

7. Find the Cholesky decomposition of the following matrix.

$$A = \begin{pmatrix} 4 & 2 & -2 & 2 \\ 2 & 26 & 13 & 1 \\ -2 & 13 & 14 & -1 \\ 2 & 1 & -1 & 10 \end{pmatrix}$$

8. Use your answer to the previous problem to solve $Ax = (1\ 2\ 3\ 4)^T$