(1) Let $U \subset \mathbb{R}^4$ be the subspace spanned by the two column vectors

$$A_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ denote the linear map given by orthogonal projection onto U. Find the matrix of P with respect to the standard basis for $\mathbb{R}^4$.

(2) Let $\langle\, , \rangle$ be a positive definite inner product on the vector space V. Let $L : V \to V$ be a linear operator that satisfies the condition

$$\langle u, L(v)\rangle = \langle L(u), v\rangle \text{ for all } u, v, \in V.$$

(Such an operator is said to be *self-adjoint.*)

Let v_λ and v_μ be eigenvectors associated to the eigenvalues λ and μ of L, with $\lambda \neq \mu$. Show that $v_\lambda \perp v_\mu$.

(3) Let V be the vector space of continuous functions on the interval $[0, \pi]$, that vanish at 0 and π; and let $\langle\, , \rangle$ be the scalar product defined by

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)\, dx.$$

Let $g_k(x) = \sin(kx)$, for $k = 1, 2, 3, \ldots$, and let $W_n \subset V$ be the subspace generated by the set $\{g_k : k = 1, 2, \ldots, n\}$.

(a) Show that $\{g_k : k = 1, 2, \ldots\}$ is an orthogonal set.

(b) Let $f(x) = x(\pi - x)$. Let f_n denote the orthogonal projection of f onto W_n. Show that

$$f_{2n+1}(x) = \frac{8}{\pi} \sum_{k=0}^{n} \frac{\sin((2k+1)x)}{(2k+1)^3}.$$

(4) Let V be the vector space of continuous functions on the closed interval $[-1, 1]$, with scalar product defined by

$$\langle f, g\rangle = \int_{-1}^{+1} f(x)g(x)\, dx.$$

(a) Apply the Gram-Schmidt orthogonalization process to the set $\{1, x, x^2, x^3\}$ to obtain an orthogonal set of four polynomials, $\{p_0(x), p_1(x), p_2(x), p_3(x)\}$.

(b) Verify that p_k is a solution of the differential equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \text{ with } \lambda = k(k+1).$$

Remark: After multiplication by constants the functions $p_k(x)$ are called *Legendre polynomials* and the differential equation is called *Legendre's equation.*