MIDTERM 1 SOLUTIONS MATH 136 SPRING 2016

1 (a)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 2 & 0 \\ 3 & 4 & 3 & 1 & 2 & 0 \\ 3 & 3 & 3 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{use row 1}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 3 & 0 & 3 & 1 & 2 & 0 \\ 3 & 0 & 3 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{use row 2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 \end{pmatrix}$$

switch rows 1 & 2 then use row 3

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

this is the row reduced echelon form.

next: to find ker L_A switch col. 3 & 4 ($\hookrightarrow S_{34}$) then switch col 6 & 4 ($\hookrightarrow S_{46}$) to get

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(b) dim ran L_A = # pivots = 4

(c) dim ker L_A = $6-4=2$

(d) to get a basis for ker L_A: ~~First note that~~

$$\begin{pmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

forms a basis for kernel of $A S_{34} S_{46}$.

Set

$$x_1 = S_{34} S_{46} \begin{pmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} = S_{34} \begin{pmatrix} -2 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}}$$

and:

$$x_2 = S_{34} S_{46} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = S_{34} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

x_1, x_2 basis for ker L_A

(e) To find a basis for range_A = col. space of A,
use the columns of A which correspond to the pivots: 1, 2, 4, 6

$$\begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2. If we expand the determinant along the first row then we obtain $Ax + By + Cz + D = 0$ for some numbers A, B, C, D which are determinants of 3×3 matrices of integers.
If we replace x, y, z by $1, 1, 1$ then two rows will be identical and hence the det. is zero.
Same thing happens for $(3, 2, 1)$ and $(1, 2, 3)$
So each of the three points satisfies the equation.
We need to show not all of $\{A, B, C, D\}$ are zero (then we have a plane)

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 1 - 3 - 1(2-2) + 1(6-2) = 2 \neq 0.$$

3. We can write

$$P v = \sum_{k=1}^{r} a_k \vec{b}_k$$

$$P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$$

so $v - Pv \in \ker P$

write: $$v - Pv = \sum_{k=r+1}^{n} a_k \vec{b}_k$$

Thus $$v = \sum_{k=1}^{n} a_k \vec{b}_k$$ so $(\vec{b}_1, \ldots, \vec{b}_n)$ span.

If $$\sum_{k=1}^{n} c_k \vec{b}_k \overset{1}{=} 0,$$ find w so that $$P(w) = \sum_{k=1}^{r} c_k \vec{b}_k$$

$$0 = P\left(\sum_1^n c_k \vec{b}_k\right) = P(P(w)) + P\left(\sum_{r+1}^{n} c_k b_k\right) = P(w) + 0 = P(w) = \sum_1^r c_k \vec{b}_k$$

so $\sum_1^r c_k \vec{b}_k = 0$. But $\vec{b}_1, \ldots, \vec{b}_r$ are indep so $0 = c_1 = \cdots = c_r$

But then by 1, $$\sum_{k=r+1}^{n} c_k \vec{b}_k = 0.$$ Since $\vec{b}_{r+1}, \ldots, \vec{b}_n$ are indep,

we must have $c_{r+1} = \cdots = c_n = 0$. We've proved $\vec{b}_1, \ldots, \vec{b}_n$

are independent. Since they also span, they form a basis.

(b) Find w_j so that $b_j = P(w_j)$ $j=1,2,3$. ($b_j \in \operatorname{ran} P$)

then $P(b_j) = P^2(w_j) = P(w_j) = b_j$

and for $j=4,5$: $P(b_j) = 0$ since $b_4, b_5 \in \ker P$

so $[P]_{\{b_1,\dots,b_5\}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

4. geom mult $\overset{\text{(given)}}{=}$ anal. mult $= n$ because λ is the only e. value. $[P_A(z) = (\lambda - z)^n]$

So A is diagonalizable to $\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} = \lambda I$.

$QAQ^{-1} = \lambda I$ so $A = Q^{-1}(\lambda I)Q = \lambda Q^{-1}Q = \lambda I$.

FOR SOME INVERTIBLE MATRIX Q.