SOLUTIONS TO MIDTERM 2 SPRING 2016 MATH 136

(THERE ARE MULTIPLE WAYS TO JUSTIFY YES/NO ANSWERS - HERE'S ONE WAY :)

1. i. $B = B^*$ yes (clearly)

ii. $B^*B = BB^*$ yes since $B^*B = B^2 = BB^*$ by ⓐ

iii. $B = SDS^{-1}$ S invertible, D diagonal - yes - all self adj. matrices are diagonalizable

iv. $B = UDU^*$ with U unitary, D diagonal yes since self-adjoint

v. Invertible: $\exists C$ so that $CB = I$. But then $(\det C)(\det B) = \det I = 1$ but $\det B = 0$ so no

vi. pos. def. $B = B^*$ and all eigenvalues are > 0

$$\det(B - \lambda I) = \lambda\left[(14-\lambda)(11-\lambda) - 4\right] \quad \text{(expand along 2nd row)}$$

$$= \lambda\left[\lambda^2 - 25\lambda + 150\right] = \lambda(\lambda - 10)(\lambda - 15)$$

so 0 is an e.value $\therefore$ no, not pos. def.

vi. $B = B^*$ and all e.values ≥ 0. Yes! - e.values are 0, 10, 15

vii. $BB^* = I$ But (1,1) entry of BB^* is $14^2 + 4 \neq 1$. No

viii. B real and $BB^T = I$ - so in (vii) the (1,1) entry of $BB^T = 14^2 + 4 \neq 1$.

NO

2. $A^*A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1+4+9 & 2+8+18 \\ 2+8+18 & 4+16+36 \end{pmatrix} = \begin{pmatrix} 14 & 28 \\ 28 & 56 \end{pmatrix}$

$\det(A^*A - \lambda I) = (14-\lambda)(56-\lambda) - 28\cdot 28 = \lambda^2 - 70\lambda + \underbrace{14\cdot 56}_{14\cdot 2\cdot 28} - 28\cdot 28$

$= \lambda^2 - 70\lambda = \lambda(\lambda - 70)$

e. values 70, 0 only non-zero eigenvalue is 70.

$(A^*A - 70I) = \begin{pmatrix} -56 & 28 \\ 28 & -14 \end{pmatrix} \xrightarrow[\text{row 1} - 2\cdot\text{row 2}]{} \begin{pmatrix} 0 & 0 \\ 28 & -14 \end{pmatrix} \xrightarrow[\text{row 2}/14]{} \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}$

The eigenspace has dimension 1, $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvalue

$v_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ is a normalized e. vector $Av_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = \sqrt{5} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$A^+ x = \dfrac{(x, Av_1) v_1}{\lambda_1}$ so $A^+ = \dfrac{v_1 Av_1^*}{70} = \dfrac{1}{70} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \sqrt{5} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \dfrac{1}{70} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

3. Extend v_1 to a basis of $\mathbb{R}^2$ by choosing $v_2 \in \ker A$

row reduce $\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in \ker A$, normalize to get $v_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

So $V = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$ Note V is unitary.

Extend $w_1 = \frac{Av_1}{\|Av_1\|}$ to a basis of $\mathbb{R}^3$ by finding an o.n. basis of $\ker A^*$

(row reduce: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\ker A^*$. Apply Gram-Schmidt to make orthogonal

$$u_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -6/5 \\ 1 \end{pmatrix}$$

So $w_1 = \frac{Av_1}{\|Av_1\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$

$$w_2 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$w_3 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{70}}\begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

$$W = \begin{pmatrix} 1/\sqrt{14} & -2/\sqrt{5} & -3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & -6/\sqrt{70} \\ 3/\sqrt{14} & 0 & 5/\sqrt{70} \end{pmatrix}$$

Note W is unitary

$$\Sigma_1' = \begin{pmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$A = W \Sigma_1' V^*$ is the singular value decomp. of A.

(EXERCISE: CHECK THIS PRODUCT!)

4. a.

$$C = \begin{pmatrix} 100 & 90 & 70 & 40 \\ 90 & \cancel{145} & 111 & 60 \\ 70 & 111 & 110 & 56 \\ 40 & 60 & 56 & 30 \end{pmatrix} \to \begin{pmatrix} 10 & 9 & 7 & 4 \\ 9 & 145-81 & 111-63 & 60-36 \\ 7 & 111-63 & 110-49 & 56-28 \\ 4 & 60-36 & 56-28 & 30-16 \end{pmatrix}$$

$$\to \begin{pmatrix} 10 & 9 & 7 & 4 \\ 9 & \sqrt{64}=8 & 6\,\cancel{48} & \cancel{24}\,3 \\ 7 & \cancel{48}\,6 & 61-36 & \cancel{28}-18 \\ 4 & \cancel{24}\,3 & \cancel{28}-18 & 14-9 \end{pmatrix} \to \begin{pmatrix} 10 & 9 & 7 & 4 \\ 9 & 8 & 6 & 3 \\ 7 & 6 & \sqrt{25}=5 & \cancel{10}\,2 \\ 4 & 3 & \cancel{10}\,2 & 5-4 \end{pmatrix}$$

So $L = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 9 & 8 & 0 & 0 \\ 7 & 6 & 5 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $\quad L^* = \begin{pmatrix} 10 & 9 & 7 & 4 \\ 0 & 8 & 6 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$C = LL^*$$

(EXERCISE: CHECK THE PRODUCT)

To solve $LL^* x = \begin{pmatrix} 0 \\ 64 \\ 43 \\ 24 \end{pmatrix}$ first solve $Ly = \begin{pmatrix} 0 \\ 64 \\ 43 \\ 24 \end{pmatrix}$

$$\begin{pmatrix} 10 & 0 & 0 & 0 \\ 9 & 8 & 0 & 0 \\ 7 & 6 & 5 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 64 \\ 43 \\ 24 \end{pmatrix}$$

$10 y_1 = 0$ so $y_1 = 0$

$9 \cdot 0 + 8 y_2 = 64$ so $y_2 = 8$

$7 \cdot 0 + 6 \cdot 8 + 5 y_3 = 43$ $\quad 5 y_3 = -5$

so $y_3 = -1$

$4 \cdot 0 + 3(8) + 2(-1) + y_4 = \cancel{24}$

so $y_4 = 2$

Next solve $L^* x = y$

$$\begin{pmatrix} 10 & 9 & 7 & 4 \\ 0 & 8 & 6 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -1 \\ 2 \end{pmatrix}$$

$\boxed{x_4 = 2}$

$5 x_3 + 2 \cdot 2 = -1$

$\boxed{x_3 = -1}$

$8 x_2 - 6 + 6 = 8$

$\boxed{x_2 = 1}$

$10 x_1 + 9 - 7 + 8 = 0$ $\quad \boxed{x_1 = -1}$

So $x = (-1, 1, -1, 2)^T$.