

Math Circle - Sets Homework

In the following problems, let A , B , and C denote sets.

- (10 points)** Using the fact that $|A \cup B| = |A| + |B| - |A \cap B|$, determine a similar formula for $|A \cup B \cup C|$ in terms of the sizes of A , B , and C and their intersections. *Bonus:* Can you come up with a general formula for any n sets A_1, A_2, \dots, A_n ?
- (10 points)** In Mr. Garfunkel's classroom, there are 100 students. He decides to conduct a simple survey. He asks everyone whether they **like** or **dislike** the three sports Basketball, Football, and Hockey. Each student can either **like** or **dislike** multiple sports and must answer for each of the three. Below are partial results. It is your job to finish off the results by answering the questions after this partial information.
 - 58 like Basketball.
 - 63 like Football.
 - 75 like Hockey.
 - 82 like at least one of Basketball or Football.
 - 87 like at least one of Basketball or Hockey.
 - 92 like at least one of Football or Hockey.
 - 1 student dislikes all three sports.
 - How many students like **both** Basketball and Football?
 - How many like **both** Basketball and Hockey?
 - How many like **both** Football and Hockey?
 - How many like **all three** sports?

3. **(10 points)** Prove the second part of De Morgan's Law: $A - (B \cap C) = (A - B) \cup (A - C)$. *Hint.* First draw a Venn diagram to get some intuition of this Law.

4. **(10 points)** The four Ninja Turtles (Leonardo, Donatello, Raphael, and Michelangelo) are arguing about who is the wisest. They seek Master Splinter's advice. "Easy," he says. "Each of you will close your eyes, and I will paint either a black stripe or a white stripe on your forehead. Then you shall open your eyes and all face each other. If any of you sees at least one white stripe, you raise your hand. The one who first guesses the color of the stripe on his forehead wins."

All the Turtles close their eyes, and Splinter paints a white stripe on all four foreheads. Upon opening their eyes, they all see at least one white stripe and immediately raise their hands as required. Hours pass as they ponder the situation in silence. Leonardo finally guesses: "I have a white stripe on my forehead." Admittedly, this is a guess – but it *is* based on some sound logical deductions. How did he come to his conclusion?

Hint. As an easier case, you could consider the case of only three Turtles.

5. **(10 points available to each team)** For finite sets, we have an intuitive understanding of what it means for one set to be *smaller* or *larger* than another. Come up with some reasons that this may not be so straightforward for infinite sets. Think hard! The more inventive the reason, the better!