Problems
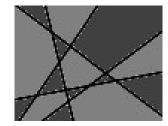
1. On a board with dimensions 5×2011 two players play the following game: on his or her turn, each player crosses out a 2×2 block of cells that does not intersect any other already crossed-out block. The player who cannot make such a move loses. Which of the two players has a playing strategy which always results in a win, and what is this strategy?

2. Start with any two-digit number. Multiply it by 3 and take the last two digits of this to get a new two-digit number. Now take this number, multiply it by 3 and take the last two digits to get yet another two-digit number, etc. (For example, starting from 13, we would have the sequence 13→39→17→51→53→59→77→…) Is there a two-digit number you can start with such that this process never brings you back to your starting number? Justify your answer.

3. Several lines divide the plane into regions. Call two regions "neighboring" if they share more than one point on a single line (i.e. share a side). Prove that it is always possible to color the regions using two colors so that no two neighboring regions are colored the same.

4. Show that among any 18 consecutive three-digit numbers it is always possible to find at least one that is divisible by the sum of its digits.

5. The Math Circle admitted 30 seventh graders. Is it possible that 9 of the students have exactly 3 friends admitted to the circle, 11 students have exactly 4 friends each, and 10 students have exactly 5 friends each?