# Math Circle - Spring 2012 - Homework 6

**1. (10 points)** A computer starts with a given positive integer to which it randomly adds either 54 or 77 every second and prints the resulting sum after each addition. For example, if the computer is given the number 1, then a possible output could be: 1, 55, 109, 186, ... Show that after finitely many seconds the computer will print a number whose last two digits are the same.

**2. (10 points)** What are the last 2 digits of $7^{(2^{100}-1)}$? *Hint.* Use the identity:

$$2^{100} - 1 = 1 + 2 + 2^2 + 2^4 + 2^8 + \cdots + 2^{99}.$$

This question can be rephrased as: *What is* $7^{(2^{100}-1)}$ *modulo* 100*?*

**3. (10 points)** Given the 400-digit number 1984198419841984...19841984, can you erase some digits from the beginning and from the end so that the sum of the remaining digits is 1984?

**4. (10 points)** An enemy Battleship team is placing their ships on a $4 \times 4$ grid as we did in the in-class worksheet. But instead of encoding their positions by multiplying by a certain $N$, the enemy is encoding their positions by first raising the position number to a certain power $M$, then reducing modulo 17.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

As an example, consider $M = 5$. Suppose the enemy has a ship at positions 3 and 7. Then they record their locations based on

$$3^M = 3^5 = 243 \equiv \mathbf{5} \pmod{17}$$

$$7^M = 7^5 = 16807 \equiv \mathbf{11} \pmod{17}.$$

The enemy will record ships at positions 5 and 11.

Suppose you know that the enemy is using $M = 7$ for encoding and is reporting a ship at positions 5, 9, and 14. Where are these ship positions actually located?