

Math Circle - Hilbert's Hotel

David Hilbert owns a hotel: *The Happening Hotel of Infinity*. This hotel is only one story tall, but has infinitely many rooms. The rooms are all down a single hallway, one after another; and the rooms are numbered, in order, 1, 2, 3, 4, Do note that Hilbert's hotel is actually located in the center of a **black hole**, so there's no real problem with infinity being "too big."

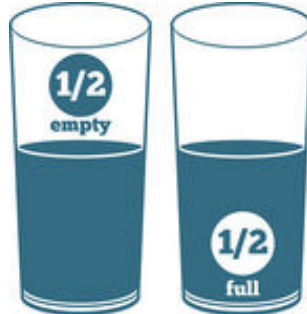
Hilbert is in need of some help. His mind is having some trouble wrapping *its* mind around the concept of infinity. He hires your team as a group of consultants to help him with the tough decisions.



1. Hilbert's hotel is all booked up for the night – there are no free rooms. However, a weary traveler from Pluto arrives at the front desk. Pleading, "Please, Good Sir Hilbert! I have been traveling for years, and I have finally arrived at your Happening Hotel! I am so tired, and I *need* a room in which to recover!"

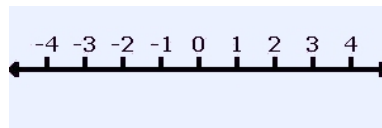
Hilbert refuses to oblige. There is no way that he is going to kick out one of his paying customers! As his consultant, can you come up with a solution which will benefit both Hilbert and the Plutonian astronaut?

2. After the weekend, Hilbert's hotel is in very sad shape. Only half of the rooms are occupied! Specifically, the hotel only has customers in the even-numbered rooms: 2, 4, 6, 8, etc.



The Intergalactic Hotel Inspector is coming *tomorrow*, and Hilbert wants his hotel to appear as full as possible for the occasion. Can you help him out?

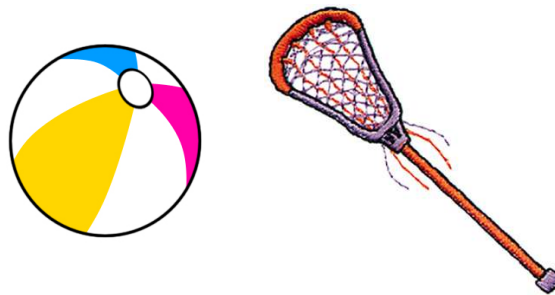
3. The Wellkits are a race of beings from a far distant **black hole**. There are an infinite number of Wellkits, each named after a (possible negative) integer. Specifically, their names are $W(1)$, $W(2)$, $W(3)$, $W(4)$, and so on. There is also a $W(-1)$, $W(-2)$, $W(-3)$, $W(-4)$, etc. There's even a $W(0)$!



The hotel is currently empty, and the Wellkits want to stay, but only if they all get their own room. Hilbert sees no way of accommodating them all – after all, there are more than *twice* as many Wellkits as there are rooms! However, the Wellkits do pay *quite* well, and so he'd like to fit as many as possible into the hotel. What is the best you can do?

4. It is the weekend of the Universal Yettelball Tournament, and Hilbert’s hotel is filled to capacity! However, just as David Hilbert is about to put out the “No Vacancy” sign, the world-famous Yettelball team – the *Infinitudes* – has come looking for some rooms. Since you were able to help out the Plutonian, Hilbert once again comes to you to try to find some space for the Infinitudes.

There is, however, just a *teensy* hiccup: there are infinitely many *Infinite* players. The players are content to be identified with their jersey numbers, which are 1, 2, 3, 4, 5, . . . , and they all want their own room. Can you do anything for Hilbert in this case?



5. After the huge fiasco that was finding hotel rooms for all of them at last year’s Tournament, the Universal Champion yettelball team from the planet Ration-al would like to reserve rooms in the *Happening Hotel of Infinity* for next year. Hilbert currently has no reservations for that week, so he doesn’t think there should be any problem giving them each a room.

However, as Hilbert is about to take their names, he realizes that there are a *lot* more Ration-al-ians than he thought! As before, they all can be identified by their jersey number, but they Ration-al jersey numbers come in *all positive rational numbers*! There is a player number $3/17$, there’s a player number $147/2281$, there’s a player number $4!$ In short, there is a Ration-al player for every single number n/m where n and m are positive integers.

Just taking $m = 1$, Hilbert sees that they already have a Ration-al player corresponding to every positive integer. This itself would fill up the hotel, not including any of the infinitely many players with denominators not equal to 1. In frustration, Hilbert screams that there’s no way he can handle all those players, and he slams down the phone. As Hilbert’s hired consultant, do you think there’s anything more that can be done for the Ration-al-ians?

6. Hilbert’s hotel is becoming quite popular. Last Sunday afternoon, Hilbert got the biggest group of aliens that he had *ever seen!* The hotel was empty, and the aliens from the planet Betridos decided that they all wanted their own room at the hotel. With your help, Hilbert was able to cope with the Wellkits and the Ration-al-ians, but these Betridosians are quite unwieldy.

To put it bluntly: the Betridosians do not have names. They have numbers instead. These numbers are not restricted to be integers or rationals – rather, the Betridosians can have *any* real number as their personal number. And indeed, it is the case that there is precisely one Betridosian for every real number. This includes a Betridosian for $147/2281$, one for -3 , and even one for π and one for $\sqrt{2}$.

Hilbert likes the Betridosians, but he *really* thinks he doesn’t have the room for them all. Of course, Hilbert’s been wrong before, and so he first decides to consult with you, the **expert** in the infinite hotel. Can you find rooms for all the Betridosians? If not, can you at least give a reason why not?

7. Hilbert decides to expand his *Happening Hotel of Infinity*. He has a lot of money saved up from his business ventures, and so he will spare no expense.

Recall that Hilbert’s hotel is only one story tall. He decides to expand by extending the hotel to **infinitely** many stories tall, each story still with **infinitely** many rooms. More precisely, now each room will be numbered with *two* integers: Room Number (n, m) , where n is the floor number and m is the corresponding room on that floor.

Hilbert’s question to you: “Does this expansion make good business sense?” Why or why not? Remember, as we’ve seen here, concepts like “bigger” and “smaller” are a bit more difficult with infinity, so justify your reasons well!

