Math Circle Induction Worksheet

November 10, 2011

Try to do these four problems by using mathematical induction.

Reminder: When proving things by induction, you have to show:

1. The **base case**, that the statement you're proving is true for some small number;

2. The **induction step**, that if the statement is true for *any* number, then it must be true for the next number.

Then you can conclude that the statement is true for all numbers from your base case upward. Happy solving!

There exists in a distant galaxy a very strange planet Щ[1]. This planet Щ is flat – it is just a circle, not a sphere like Earth, and all the Щlings live on one side.

1. On the surface of planet Щ, someone has drawn several mysterious circles hundreds of miles across, thus dividing it into many countries.
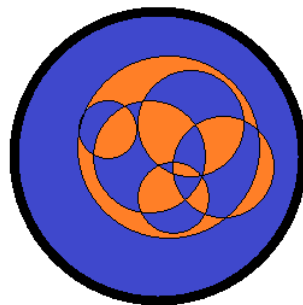
   In each country, the local aliens are either blue or red. Show that the tribes of aliens in each country can choose colors in such a way that no two bordering countries have aliens of the same color.

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[1]If you're wondering, the natives pronounce it [ɕɕa] ("Shcha")

2. Your typical evil space ninja wants to destroy the planet by cutting it into pieces with his laser. However, his laser can only give 13 straight laser bursts before running out of ψ-particles. If the planet breaks into more than 80 pieces, it will collapse on itself and fall into the sun. But if the planet is cut into 80 pieces or fewer, then the pieces will eventually gravitate back together again. Will the ninja be able to destroy the planet? *(Hint: Find, by induction, the number of pieces that can be made with n laser bursts.)*
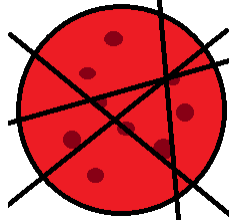
3. Since our planet Щ has been cut into pieces, each sector now prints its own currency. Sector A prints money in ₤7 amounts, Sector B in ₤3 amounts, and Sector C doesn't print money at all. A traveler to Sector C brings only money printed in Sectors A and B. Using exact cash only, the traveler cannot purchase an item costing ₤2 or less! What are *all* the values of items that the traveler is unable to purchase?

   For example, he can get an electronic brain that costs ₤37 by paying with 10 ₤3 bills and a ₤7 bill, but there is no possible way to get a spaceship paint job that costs ₤9.

   (Hint: Remember the squares problem?)

4. **Challenge:** After the planet gravitates back into a circle, a group of intrepid scientists from a neighboring planet decides to launch a mission to explore the circumference (the outer edge) of Щ. They plan to fly around the edge of the planet and drop packets of fuel at regular intervals, and then place a rover on the planet that will traverse the edge, picking up fuel as it goes. However, the evil space ninja hacks the scientists' program, and everything goes wrong!

   Not only does the fuel not get dropped at regular intervals, it also doesn't get dropped in equal amounts! All the scientists know is that the total amount of fuel that was dropped is exactly enough for the rover to make one trip around the circumference of Щ. Show that there exists a place along the edge where they can drop the rover so that it can travel the whole way around the edge of the planet.