Math Circle - Relations

A **(binary) relation** $\propto$ can be defined on any set S. For any points $x, y \in S$, either x satisfies the relation with y, or not. In the former case we write $x \propto y$.

Examples.

Some example sets	
$\mathbb{Z}$	= integers
$\mathbb{R}$	= real numbers
P	= all people

S	$\mathbf{x} \propto \mathbf{y}$
$\mathbb{Z}$	$x \leq y$
$\mathbb{Z}$	$x > y$
$\mathbb{Z}$	$x = 4$
$\mathbb{Z}$	$x - y$ is divisible by 5
$\mathbb{R}$	$x - y$ is an integer
$\mathbb{R}$	x is an integer or y is irrational
P	x and y have the same gender
P	x has more siblings than y
P	x lives next door to y
P	x is the brother of y

The following is a list of different types of relations.

- **Symmetric:** If $a \propto b$, then $b \propto a$.
- **Transitive:** If $a \propto b$ and $b \propto c$, then $a \propto c$.
- **Reflexive:** For all $a \in S$, $a \propto a$.
- **Antisymmetric:** For all *distinct* $a, b \in S$, if $a \propto b$, then $b \not\propto a$.
- **Total:** For all $a, b \in S$, it must hold that $a \propto b$ or $b \propto a$ (or both).

For each of the example relations described above, determine which of the five properties they satisfy.

If you can, give an example of a relation on P satisfying each of the following. Try to come up with examples which satisfy *only* the specified conditions.

Symmetric	Transitive	Reflexive	Antisymmetric	Total
—	—	—	—	—
✓	—	—	—	—
—	✓	—	—	—
—	—	✓	—	—
—	—	—	✓	—
—	—	—	—	✓
✓	✓	—	—	—
✓	—	✓	—	—
✓	—	—	✓	—
✓	—	—	—	✓
—	✓	✓	—	—
—	✓	—	✓	—
—	✓	—	—	✓
—	—	✓	✓	—
—	—	✓	—	✓
—	—	—	✓	✓
✓	✓	✓	—	—

Definition. A relation which is symmetric, transitive, and reflexive is called an **equivalence relation**.

Give as many examples as you can of equivalence relations on C, where C is the set of all students in the class.

Can you say anything about how an equivalence relation "breaks up" the class?

Why do the following definitions sound like they make sense?

Definition. A relation which is transitive, reflexive, and antisymmetric is called a **partial order**.

Definition. A partial order which is also a total relation is called a **total order**.