Math Circle - Bijective Proofs

In combinatorics, it is often the case that we can prove an equation is true by means of some really tedious algebraic manipulation of both sides of the equality. This has two major downsides: 1) it is quite easy to make small algebra errors when moving around factorials and adding fractions – *chooses* and *factorials* are **rarely** fun to work with algebraically. And 2) such a solution offers little insight into the reason *why* the formula is true.

In contrast, the preferred method of proving the validity of a formula is called a **bijective proof**. This method of proof is ideal for situations in which you want to prove the truthiness of an equation of the form

$$\text{expression } \#1 = \text{expression } \#2.$$

In such a case, the step-by-step description of a bijective proof is:

1. Identify a quantity #1 which expression #1 counts.
2. Identify a quantity #2 which expression #2 counts.
3. Show that quantity #1 = quantity #2.

We say that have found a **bijection** between the two quantities – that is, a one-to-one correspondence between them. It can save you some work if you can make it so that quantities #1 and #2 are actually exactly the same!

Example. Prove that $\binom{n}{k} = \binom{n}{n-k}$. Verify this for $n = 5$ and $k = 2$.

Solution. Recall that $\binom{n}{k}$ denotes the number of ways of choosing k balls from a collection of n distinguishable balls, where the order of your choice does not matter. In a similar fashion we know that $\binom{n}{n-k}$ denotes the number of ways of forming a collection of $n - k$ balls from the total n.

But forming a collection of k balls is the same as *not forming* a collection with the other $n - k$ balls. In other words, the number of ways of forming collections of k balls is exactly the same as the number of ways of *not forming* collections of $n - k$ balls. Hence, the two expressions are the same, which finishes the proof. $\square$

Give a bijective proof of each of the following formulas. In each problem it is assumed that $n \geq k \geq 0$. Numerically verify each formula for some small n and k.

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$

   Hint. Show that both expressions count the number of subsets of $\{1, 2, \cdots, n\}$.

2. $\binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k} = \binom{n+1}{k+1}.$

   Hint. The number on the right represents the number of ways of choosing $k+1$ balls from a collection of $n+1$ distinguishable balls.

3. $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3.$

   Hint. The number on the left is the number of triples of the form (a, b, c), where a, b, and c can be any positive integer $1, 2, \ldots, n$.

4. A **composition of** n is a way of writing n as a sum of positive integers,

   $$n = a_1 + \cdots + a_k.$$

   Different orders count as different compositions of n. Let $c(n)$ denote the total number of compositions of n. For example, $c(4) = 8$ because the 8 compositions of 4 are:

   $1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 1+3, \quad 3+1, \quad 2+2, \quad 4.$

   Prove that $c(n) = 2^{n-1}$ by showing that $c(n)$ is the same as the number of subsets of a set with $n-1$ elements.