Math Circle - Fibonacci Equations

Let f_n denote the number of ways of tiling a $1 \times n$ board with any number of 1×2 dominoes for $n \geq 0$. It is not too hard to show that

$$f_n = F_{n+1},$$

where F_k denotes the kth Fibonacci number. Recall, $F_0 = 0$, $F_1 = 1$, and for $k \geq 2$

$$F_k = F_{k-1} + F_{k-2}$$

Notice that the f_n satisfy this same recurrence relation. The first few values of f_n:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad \ldots$$

In the following exercises Cassini and Catalan are mathematicians who are obsessed with the Fibonacci numbers. You may solve the problems by using bijective proofs, induction, or by making use of the recurrence relation describing f_n.

1. Cassini and Catalan have just won the lottery, and they want to give you — their loyal friend — a whole bunch of poker chips. Cassini and Catalan have chosen in secret some really large value of N, and they give you a choice.

   - Cassini will give you $f_0 + f_1 + f_2 + \cdots + f_N$ poker chips.
   - Catalan wants to give you f_{N+2} poker chips.

   Who do you choose to go with in order to get the most value? Does the value of N matter?

2. After failing to provide you with the poker money they promised in the last problem, Cassini and Catalan have agreed to give you some real cash instead. For another large value of N:

   - Cassini will give you f_{N-1} dollars every hour for a total of f_{N+1} hours.
   - Catalan wants to give you f_N dollars every hour for a total of f_N hours.

   Who do you choose to get the most money? Does the value of N matter?

3. Catalan rushes in from the next office over. "Cassini! Cassini!" he shouts, waving a sheet of paper through the air. "Old Fibonacci was incorrect! The number of ways of tiling a $1 \times n$ board with 1×2 dominoes is actually

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots$$

   I have a proof, but it's a bit too large to fit in the margin of this page."

   Is Catalan correct in making such a quick judgement of poor Fibonacci?

4. Cassini is tired of Catalan's naysaying. To keep him occupied and out of the way, Cassini tells Catalan that he will give him a number of Oreo cookies equal to the smallest positive integer which cannot be written as the sum of **distinct** Fibonacci numbers.

   "Pssh!" Catalan waves him off. "How big can the smallest such number possibly be?" After a few minutes, curiosity gets the best of him and Catalan begins testing some small integers. He quickly becomes convinced that Cassini is going to owe him a pretty hefty lot of Oreos.

   Should Cassini be worried about having to pay up big?