Math Circle - Invariants

1. Step right up to Chris and Kolya's Amazing Magic Show! We're going to let you in on a few of the tricks which we have up our sleeves. For each of these tricks, you first start with the numbers $1, 2, \ldots, 10$ written on the board. In each case, you are permitted to come up and erase any two numbers a and b, and then write the new number indicated. For each problem, perform that action exactly nine times, and you'll end up with a single number on the board. Can you determine what this final number will be in each of the problems? Better, can you prove that why it will always be the same, independent of your sequence of nine choices?

   (a) Replace with $a + b + ab$.

   (b) Replace with $\dfrac{ab}{a+b}$.

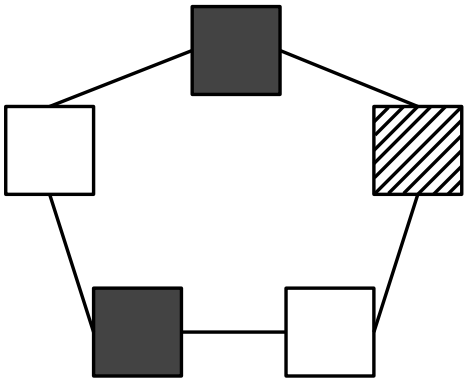
   (c) Replace with $\min(a, b)$, i.e. the smallest of a and b.

2. You have received a secret message. It is encoded in a lock which has 5 keys arranged in a pentagon. Each key is a square which is filled as either solid, striped, or empty. In order to open the lock, every key must have the same filling.

   A key can be switched — this means that the filling is switched as follows:

   $$\text{solid} \to \text{striped} \qquad \text{striped} \to \text{empty} \qquad \text{empty} \to \text{solid}$$

   Each time you choose to switch a key you must also switch the two keys immediately adjacent to it. Assuming the starting position below, is it possible to determine any sequence of switches which will unlock the secret message?



3. You have received another secret message. This time the lock has 16 keys arranged in a 4×4 array. Each key is an arrow pointing in one of the four cardinal directions (N, E, S, W). In order to open the lock, each key must be oriented in the same direction.

   A key can be switched — this means that the arrow is rotated 90° clockwise. Except you cannot switch a single key at a time, you must switch either an entire row or an entire column at once.

   Assuming the starting position below, is it possible to determine a sequence of switches which will unlock the secret message?