Vieta's Formulas and the Identity Theorem

This worksheet will work through the material from our class on 3/21/2013 with some examples that should help you with the homework. The topic of our discussion will be **polynomials**. The first thing we'll do is define a polynomial:

Definition: A polynomial is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0$$

where x is a variable (you can plug in anything you want for x) and the $a_0, a_1, a_2, \ldots a_n$ are all constants. We call these constants the **coefficients** of the polynomial. We call the highest exponent of x the **degree** of the polynomial.

Example:
This is a polynomial:

$$P(x) = 5x^3 + 4x^2 - 2x + 1$$

The highest exponent of x is 3, so the degree is 3. $P(x)$ has coefficients

$$a_3 = 5$$
$$a_2 = 4$$
$$a_1 = -2$$
$$a_0 = 1$$

Since x is a variable, I can *evaluate the polynomial* for some values of x. All this means is that I can plug in different values for x so that my polynomial simplifies to a single number:

$$P(1) = 5 \cdot (1)^3 + 4 \cdot (1)^2 - 2 \cdot (1) + 1 = 8$$
$$P(0) = 5 \cdot (0)^3 + 4 \cdot (0)^2 - 2 \cdot (0) + 1 = 1$$
$$P(-1) = 5 \cdot (-1)^3 + 4 \cdot (-1)^2 - 2 \cdot (-1) + 1 = 2$$

Practice Problem:
For practice, let's analyze the polynomial $P(x) = x^3 + x - 2$. Write down all the coefficients of $P(x)$, find the degree of $P(x)$, and evaluate $P(-1), P(2)$, and $P(1)$.

Solution: $a_0 = -2, a_1 = 1, a_2 = 0, a_3 = 1$, $P(-1) = -4$, $P(2) = 8$, $P(1) = 0$.
The degree is 3.

Notice that in the practice problem above, $P(1) = 0$. Values at which complicated polynomials simplify to zero are very important and are known as the **roots** of the polynomial.

Definition: The roots of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

are the values $r_1, r_2, r_3, \cdots$ such that

$$\begin{aligned} P(r_1) &= 0 \\ P(r_2) &= 0 \\ P(r_3) &= 0 \\ &\vdots \end{aligned}$$

Example:
The roots of the polynomial

$$P(x) = x^3 + 7x^2 + 6x$$

are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -6 \\ r_3 &= 0 \end{aligned}$$

You should check that $P(-1) = 0$, $P(-6) = 0$, and that $P(0) = 0$.

Practice Problem:
Find two roots of the polynomial $P(x) = x^2 - 1$. (Try guessing some small numbers). What is the degree of $P(x)$?

Solution: $r_1 = 1, r_2 = -1$. The degree is 2.

Perhaps you have noticed that in the last two examples the number of roots is the same as the degree of the polynomial. This is not just a coincidence - there is a theorem that says that this will always be true:

Theorem 1: A polynomial of degree n has exactly n roots. **However**, some of the roots may be very complicated (some may be *complex numbers*). Don't be discouraged if you cannot immediately find all n roots.

Example:
The polynomial $P(x) = 5x^3 + 4x^2 - 2x + 1$ has degree 3 ($n = 3$), so it has exactly three roots: r_1, r_2, and r_3. However, finding these roots would be *really* hard (in fact, two of them are complex and the other one is very messy!)

Although many polynomials have very complicated roots, we can often determine a lot about them using **Vieta's Formulas**, the first of which we look at next:

Theorem 2: Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$, with roots $r_1, r_2, r_3, \ldots r_n$, the sum of the roots is given by

$$r_1 + r_2 + r_3 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$$

Example: Let's look at the polynomial $P(x) = 2x^4 - 6x^3 + 5x^2 + x - 3$. Notice that this polynomial has degree 4 (so $n = 4$), so by Theorem 1 we know that it has exactly 4 roots: call them r_1, r_2, r_3, r_4. Without knowing anything else about $P(x)$, Theorem 2 gives us a very easy way to find the sum of the roots:

$$r_1 + r_2 + r_3 + r_4 = -\frac{a_3}{a_4} = -\frac{-6}{2} = 3$$

Practice: Find the sum of the roots of the polynomials $P(x) = 3x^3 + 2x^2 - 6x - 3$ and $G(x) = x^2 - 1$.

Solution: $P(x)$ has degree 3 ($n = 3$), so the sum of the three roots of $P(x)$ is $-\frac{a_2}{a_3} = -\frac{2}{3}$.
$G(x)$ has degree 2 ($n = 2$), so the sum of the three roots of $G(x)$ is $-\frac{a_1}{a_2} = -\frac{0}{1} = 0$.

Hopefully you find it really cool that just by looking at a polynomial you can determine the sum of its roots. But we're not done quite yet. Vieta's Formulas are a set of n equations, where sum of the roots is just the first one.

Theorem 3 (Vieta's Formulas): Consider the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ with degree n. By Theorem 1 $P(x)$ has n roots, call them $r_1, r_2, \ldots r_n$. Vieta's Formulas say that

$$r_1 + r_2 + r_3 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$$

$$(r_1r_2 + r_1r_3 + r_1r_4 + \ldots r_1r_n) + (r_2r_3 + r_2r_4 + \ldots + r_2r_n) + \ldots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}$$

$$(r_1r_2r_3 + r_1r_2r_4 + r_1r_2r_5 + \ldots r_1r_2r_n) + (r_1r_3r_4 + r_1r_3r_5 + \ldots + r_1r_3r_n) + \ldots + r_{n-2}r_{n-1}r_n = -\frac{a_{n-3}}{a_n}$$

$$\vdots$$

$$r_1r_2r_3r_4 \ldots r_n = (-1)^n \frac{a_0}{a_n}$$

These formulas look quite complicated - below you'll find a simplified version of them. But before before you rush to see the Simplified Vieta's Formulas, let's notice some patterns. The first equation we are already comfortable with: all it says is that the sum of the roots is $-\frac{a_{n-1}}{a_n}$. Now let's look at the second equation. The left hand side is a sum of *every possible product of a pair of roots*. The first parentheses has every possible pair that involves r_1, the second has every possible pair that involves r_2, etc. Now let's move on to the third equation. Now the left hand side is a sum of *every possible product of three roots* (the parentheses are organized similarly). Hopefully the pattern is clear now, so it makes sense that the last equation is a product of all the roots (*every possible sum of n roots*). The right hand side is a little bit easier: it is always of the form $\pm\frac{a_{\#}}{a_n}$, where the sign ($\pm$) alternates with every equation. If the degree (n) is even, then the last equation ends with a positive sign and if n is odd then it ends on a negative sign. That's why we have the $(-1)^n$ in the last equation.

Now let's look at a **simpler version** of Vieta's Formulas...

Simplified Vieta's Formulas: In the case of a polynomial with degree 3, Vieta's Formulas become very simple. Given a polynomial $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with roots r_1, r_2, r_3, Vieta's formulas are

$$
\begin{aligned}
r_1 + r_2 + r_3 &= -\frac{a_2}{a_3} \\
r_1r_2 + r_1r_3 + r_2r_3 &= \frac{a_1}{a_3} \\
r_1r_2r_3 &= -\frac{a_0}{a_3}
\end{aligned}
$$

Example: Find the sum of the roots and the product of the roots of the polynomial $P(x) = 3x^3 + 2x^2 - x + 5$.

By the first of the Simplified Vieta's Formulas, the sum of the roots is

$$r_1 + r_2 + r_3 = -\frac{a_2}{a_3} = -\frac{2}{3}$$

and the product is

$$r_1r_2r_3 = -\frac{a_0}{a_3} = -\frac{5}{3}$$

Practice: Find the roots of the polynomial $P(x) = x^3 - x$ using Vieta's Formulas. (Hint: first figure out what the product of the roots must be. Then, figure out what the sum of the roots must be. Then you should be able to guess what the roots are).

Solution: Applying the Simplified Vieta's Formulas we find that the sum of the roots and the product of the roots must be 0. Since the product of the roots is zero, at least one of the roots must be zero. Since the sum of the roots is zero, the other two roots must be negatives of each other (or they are all zero). Now you can either plug all of this into the last of Vieta's Formula, or you can guess that the roots are $1, -1$, and 0.

The last topic that we covered (very quickly) was the **Identity Theorem**. This theorem tells us when two polynomials are actually the same polynomial.

Theorem 4 (Identity Theorem): Suppose we have two polynomials: $P(x)$ and $G(x)$, and both of them have degree less than n. Suppose there are n values $\alpha_1, \alpha_2, \ldots \alpha_n$ such that

$$
\begin{aligned}
P(\alpha_1) &= G(\alpha_1) \\
P(\alpha_2) &= G(\alpha_2) \\
&\vdots \\
P(\alpha_n) &= G(\alpha_n)
\end{aligned}
$$

(This just means that $P(x)$ and $G(x)$ evaluate to the same thing for n different values of x.) Then $P(x)$ and $G(x)$ are the same polynomial.

Example: Suppose I give you two polynomials: $P(x) = ax^2 + bx + c$ and $G(x) = Ax^2 + Bx + C$. All you know about them is that $P(1) = G(1)$, $P(5) = G(5)$, and that $P(-1) = G(-1)$. Then **you** can guarantee that $P(x)$ and $G(x)$ are actually *the same polynomial* (so $a = A$, $b = B$, and $c = C$). Indeed, since both $P(x)$ and $G(x)$ have degree less than 3 (both have degree two), the Identity Theorem tells us that if P and G evaluate to the same thing at three values of x then they are the same polynomial. Well, I told you that they evaluate to the same thing at 1, 2, and 5, so they must be the same polynomial.

Practice: You are given a polynomial $P(x)$ that has degree less than 10. Suppose that $P(x)$ has at least 10 roots (ie, $P(x) = 0$ for at least 10 different values of x). Prove that $P(x)$ is always zero.

Solution: Define the polynomial $G(x)$ so that it is zero everywhere. We can write this as $G(x) = 0$ (so *all* of the coefficients are 0, and the degree is 0). I will use the Identity Theorem to prove that $G(x)$ is the same polynomial as $P(x)$, which would mean that $P(x)$ is zero everywhere (always zero). Notice that $P(x)$ is zero at at least 10 values of x, so $P(x)$ evaluates to the same thing as $G(x)$ at these 10 values of x (because $G(x)$ is *always* zero). Since the degrees of $P(x)$ and $G(x)$ are both less than 10, by the Identity Theorem this means that $P(x)$ and $G(x)$ are the same polynomial. Therefore $P(x)$ is everywhere zero.

That's it! Congratulations on reaching the end! Shoot me an email to let me know that you've gone through this worksheet and I'll bring you some candy next week!