Math Circle - Combinatorics

Definition. The notation $\binom{n}{k}$ means the number of ways of choosing k things (such as socks) from a collection of n things, where order doesn't matter.

You may have seen the ${}_nC_k$ notation for this before, or even seen formulas for calculating it. We're going to try to forget all those formulas, and just write everything in terms of this notation.

Example. Calculate $\binom{n}{k}$ for some specific values of k:

(a) $\binom{n}{0} =$

(b) $\binom{n}{1} =$

(c) $\binom{n}{n} =$

We have a couple of neat theorems about this notation. These theorems are proved using a technique called *bijective proofs*, where we show that two numbers are equal by showing that they both count the same number of things.

Theorem 1. $n^2 = \binom{n}{1} + 2\binom{n}{2}$.

Proof: Suppose we have a drawer filled with n different socks. Let's count the number of ways that I can pull one sock from the drawer, put it back into the drawer, and then draw a second sock from the drawer.

On one hand, the number of ways of doing this is $n \cdot n = n^2$. This is because we have n choices for the first sock, and then another n choices for the second sock (since we put the first sock back).

On the other hand, let's count this a different way. There are two different cases to consider. Adding the two numbers from the different cases together gives the righthand side of the equation:

Case 1. *The same sock is drawn both times.* The number of ways to do this is the same as the number of ways of choosing a single sock from the drawer: $\binom{n}{1}$.

Case 2. *The two drawn socks are different.* For this case, there are $\binom{n}{2}$ ways of choosing which two socks you will draw. For each choice of two socks, there are two ways to draw them, since you could draw them in either order. Hence, there are $2\binom{n}{2}$ total possibilities for this case. □

Theorem 2. $\binom{n}{k} = \binom{n}{n-k}$.

Proof: $\binom{n}{k}$ denotes the number of ways of choosing k socks from a collection of n socks. Think of this as the number of ways to pull k socks at once out of a drawer filled with n socks. We want to show that $\binom{n}{n-k}$ somehow counts the exact same thing.

Taking k socks out of the drawer is the same as leaving $n-k$ socks in the drawer. Hence, the number of ways of pulling k socks out of the drawer is the same as the number of ways of choosing $n-k$ socks to leave in the drawer. □

Theorem 3. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: $\binom{n+1}{k}$ denotes the number of ways of taking k socks out of a drawer filled with $n+1$ socks. To show that this is the exact same number of things counted by the righthand side of the equation, first put all our $n+1$ socks in a row. We have two cases here, when pulling k socks out of the drawer.

Case 1. *We do not take the first sock.* In order to get our choice of k socks, we have to choose them all from the remaining n socks in the row. There are $\binom{n}{k}$ ways of doing this.

Case 2. *We do take the first sock.* Now, in order to get our choice of k socks, we have to choose $k-1$ from the remaining n socks in the row. There are $\binom{n}{k-1}$ ways of doing this.

Adding the two numbers of choices from cases 1 and 2 together, this is the righthand side of the equation. □