Problem Set 13

UW Math Circle – Advanced Group

Session 19 (6 March 2014)

1. (a) Use Lagrange's theorem to prove Fermat's Little theorem: if a is not divisible by a prime p, then $a^{p-1} \equiv 1 \pmod{p}$. *(Hint: consider the set* $\{1, 2, 3, \ldots, p-1\}$ *with the operation of multiplication modulo p.)*

   (b) Prove Euler's theorem, a generalisation of Fermat's Little theorem: if a and n are relatively prime, then $a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n)$ is the number of integers less than n and relatively prime to n.

   For example, $\varphi(10) = 4$ because $1, 3, 7, 9$ are relatively prime to 10. By Euler's theorem, this shows that a^5 has the same last digit as a for all positive integers a.

2. A fleet of 9 starships is again approaching a Martian spaceport in a line. The are able to perform two moves: (1 5)(2)(3 9 8)(4 6 7) and (1 7 8 3 5)(2)(4)(6 9). The fleet commander claims that using these moves the ships can get into 4620 possible orders. Prove that the commander is lying or made an error in his calculations.

3. A 4×100 board is covered with 200 domino tiles. Prove that it is possible to divide the board into two parts with a straight cut without cutting through any dominoes.