

Problem Set 13

UW Math Circle – Advanced Group

Session 19 (6 March 2014)

- (a) Use Lagrange's theorem to prove Fermat's Little theorem: if a is not divisible by a prime p , then $a^{p-1} \equiv 1 \pmod{p}$. (*Hint: consider the set $\{1, 2, 3, \dots, p-1\}$ with the operation of multiplication modulo p .*)
(b) Prove Euler's theorem, a generalisation of Fermat's Little theorem: if a and n are relatively prime, then $a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n)$ is the number of integers less than n and relatively prime to n .

For example, $\varphi(10) = 4$ because 1, 3, 7, 9 are relatively prime to 10. By Euler's theorem, this shows that a^5 has the same last digit as a for all positive integers a .

- A fleet of 9 starships is again approaching a Martian spaceport in a line. They are able to perform two moves: (1 5)(2 3 9 8)(4 6 7) and (1 7 8 3 5)(2)(4)(6 9). The fleet commander claims that using these moves the ships can get into 4620 possible orders. Prove that the commander is lying or made an error in his calculations.
- A 4×100 board is covered with 200 domino tiles. Prove that it is possible to divide the board into two parts with a straight cut without cutting through any dominoes.

The image shows a musical score for piano, consisting of three systems of music. The first system is marked *più lento* and *una corda*, with dynamics *pp* and *legato*. The second system is marked *Tempo I* and *tre corde*, with dynamics *rit.*, *espress.*, *cresc.*, and *pp*. The third system continues the piece with various rhythmic patterns and dynamics.