

Problem Set 6

UW Math Circle – Advanced Group

Session 8 (14 November 2013)

1. (a) Scientists have designed a very strange calculator. It has a screen and four buttons: $0, +, -, \div$.
When you press 0 , the number of the screen is reset to 0 .
When you press $+$, the number on the screen is increased by 1 .
When you press $-$, the number on the screen (x) is replaced by $-x$.
When you press \div , the number on the screen (x) is replaced by $\frac{1}{x}$. Do not even try to press $/$ when the number on the screen is 0 .
Prove that you can enter any rational number on this calculator. (*Hint: Try induction on the denominator.*)
(b) The $-$ and \div buttons have gotten jammed. You can press both of them together, but not separately. For example, from 5 you could get $\frac{-1}{5}$. Prove that you can still enter any rational number on this calculator.
2. Let C and D be two disjoint compact sets in the plane. Show that there is a positive number m such that $d(c, d) > m$ for all $c \in C$ and $d \in D$. (The “largest” such m is called the *distance* between C and D .)
3. Every point in the plane is coloured green or blue.
 - (a) Prove that there is a segment of length 1 whose endpoints have the same colour.
 - (b) Prove that there is an equilateral triangle whose vertices have the same colour.

