Problem Set 11 Solutions

UW Math Circle – Advanced Group

Session 17 (13 February 2014)

1. See the official solution, `http://www.bamo.org/attachments/bamo2008examsol.pdf`.

2. The order of σ is the least common multiple of the lengths of its cycles. Since $\sigma \in S_n$, all of its cycles have length no more than n. Clearly the least common multiple of some numbers not greater than n is not more than $n!$.

3. Let $a = (1\,2)$ and $b = (2\,3\,4\,\ldots\,100\,1)$. Notice that $b^{99}ab = (2\,3)$, $b^{98}ab^2 = (3\,4)$, etc. Thus we can perform any transposition of two adjacent elements. As we showed, this means we can make any permutation.

4. Equivalently, we must find how many transpositions is takes to get from a list of the numbers $1 \ldots n$ (in some order) to a correctly sorted list.

   We show by induction that at most $\binom{n}{2}$ transpositions are needed. Base case $n = 2$ is obvious, so suppose we know it takes $\binom{k}{2}$ moves to do this for k.

   Now consider a permutation of $k+1$ numbers. We can move $k+1$ to the end of the list with no more than k transpositions; then, by the induction hypothesis, we can do $\binom{k}{2}$ transpositions to order the first k elements in the list without touching $k+1$. Thus we have made no more than $\binom{k}{2} + k = \binom{k+1}{2}$ transpositions.

   Note that the "worst case" is the reversal permutation $[n\,n-1\,n-2\,\ldots\,3\,2\,1]$. There are $\binom{n}{2}$ pairs of numbers that are not in the correct order, and each transposition decreases this quantity by at most 1.