

Problem Set 1 Solutions

UW Math Circle – Advanced Group

Session 3 (10 October 2013)

1. Suppose we are given a convex pancake. First, notice that if we are told that the cut must begin at a certain point (X) on the boundary, there is an “opposite” point ($p(X)$) on the boundary such that the cut from X to $p(X)$ cuts the pancake into two pieces of equal perimeter. We showed this last week: we start by placing $p(X)$ very close to X on the left and move it around the edge until it comes back to X from the right and use continuity.

To create two pieces of equal area, we are going to move X . Start by placing X anywhere on the edge (call this starting point X_0) and draw the cut from X to $p(X)$. If this cut also makes two pieces of equal area, we are done. Else, suppose, without loss of generality, that the left half has larger area than the right half. Move X counterclockwise along the edge until it reaches $p(X_0)$, where $p(X)$ began. Now $p(X) = X_0$ and the half to the left of the cut is the part that used to be to the right of the cut. Therefore, the right half now has larger area than the left half. It follows by continuity that at some point in between the part to the left of the cut and the part to the right of the cut had the same area.

2. Let us count the number of pairs of neighboring states which are ruled by different parties. Every time there is a change in the ruling party of some state, this quantity strictly decreases. This number could not decrease forever (because it could not be negative), so eventually there will be no more changes in the ruling parties.
3. Let us count the number of Star Wars stories in the first 12 of the 24 hours (0:00 to 12:00). Call it a . Then count the number of Star Wars stories in the last 12 of the 24 hours (12:00 to 24:00). This is $12 - a$.

Select the block from 0:00 to 12:00 and repeatedly move it forward by one hour until you get the block from 12:00 to 24:00. Every time the block is moved forward by an hour, the number of Star Wars stories in the block changes by at most 1. Now, if $a \geq 6$, then $12 - a \leq 6$, so at some block in between there were exactly 6 Star Wars stories. An identical argument applies if $a \leq 6$. So, there is a block with 6 Star Wars (and 6 Star Trek) stories.

4. When the weeds spread to a new plot, the perimeter of the region infested by weeds does not increase. The perimeter at the beginning is at most 36 squares; if the weeds spread to the whole field, the perimeter should be 40 squares, which is impossible.

Conjecture: The greatest possible area to which the weeds can spread is 81 squares. Can you prove it?

5. Consider the quantity

(number of people facing me from the left) – (number of people facing me from the right).

When I move past one person to the right, this quantity either decreases by 1 (if I go past someone facing left or right) or does not change (if I go past someone facing forward). At the left end of the line, this quantity is nonpositive because no people are to the left. At the far right, it is nonnegative because no people are to the right. So, at some point in between it is 0.