Problem Set 4 Solutions

UW Math Circle – Advanced Group

Session 7 (7 November 2013)

1. Suppose $\alpha \neq 1$. Choose any y and let x be such that $\alpha(x+y) = x$, that is, $x = \frac{\alpha y}{1-\alpha}$. Then $f(x) + f(y) = f(\alpha(x+y)) = f(x)$, so $f(y) = 0$. Because y was arbitrary, f is constantly 0.

   If $\alpha = 1$, we can take f to be the identity function, for example.

2. If we denote the second number by x, we can compute all the numbers:

   $$1 \qquad x \qquad x-1 \qquad -1 \qquad -x \qquad 1-x \qquad 1 \qquad x \qquad \ldots$$

   Because each number depends only on the two previous ones and 1 x has already been encountered, the sequence will repeat with period 6. We find $2014 \equiv 4 \pmod 6$, so the 2014th number is the same as the 4th, that is, -1.

3. It is easy to show that $f(nx) = nf(x)$ for any integer n. Now,

   $$f(x) = f\left(n\frac{x}{n}\right) = nf\left(\frac{x}{n}\right),$$

   so $f\left(\frac{x}{n}\right) = \frac{f(x)}{n}$.

   For any rational $\frac{p}{q}$, $f\left(\frac{p}{q}\right) = \frac{f(p)}{q} = \frac{pf(1)}{q}$. Let $a = f(1)$.

4. We call a cow *even* if its weight is even and *odd* if its weight is odd.

   Notice that either all cows are even or all cows are odd. Indeed, suppose there is an even cow C and an odd cow D. Remove C from the herd and find the parity of the weight of the remaining 32 cows. Do the same with D. In one of the two cases, the weight of the remaining cows will be odd, so they could not be split in half.

   Now we will construct a herd where every cow weighs strictly less than in the current herd.

   If all cows are even, divide the weight of each cow in half. The new herd still has the property that any subherd of 32 cows can be split in half.

   If all cows are odd, subtract 1 from the weight of each cow. The new herd still has this property (recall that the two groups into which we split 32-cow subherds must each contain exactly 16 cows).

   Notice that these operations could never lead to a herd containing a cow with negative weight. These operations strictly reduce the weight of the entire herd until we reach a herd where every cow has weight 0.

   We can now reverse all the above operations and find that in the original herd all the cows had the same weight.