Extreme values, semi-invariants and continuity

UW Math Circle – Advanced Group

Session 2 (3 October 2013)

1. 100 numbers are placed on the vertices of a 100-gon such that each number is equal to the mean of its neighbors. Prove that all 100 numbers are equal.
2. Numbers are placed in the cells of a 4×4 chessboard such that each number is equal to the mean of its neighbors. Prove that all 100 numbers are equal.
3. Several identical coins are placed on a table. Prove that there is at least one coin that is touching no more than three others.
4. Several coins of different sizes are placed on a table. Prove that there is at least one coin that is touching no more than five others.
5. 1000 points are marked in the plane. Prove that you can draw a line such there are 500 marked points on each side of the line.
6. Given a convex pancake, prove that you can cut it into two pieces of (a) equal area, (b) equal perimeter.
7. (MHO 2012) Harry has an 8×8 board filled with the numbers 1 and -1, and the sum of all 64 numbers is 0. A magical cut of this board is a way of cutting it into two pieces so that the sum of the numbers in each piece is also 0. The pieces should not have any holes. Prove that Harry will always be able to find a magical cut of his board.
8. There are several cities in a certain kingdom. An obnoxious citizen is exiled from city A to city B, which is the farthest city in the kingdom from A. After a while, he is again exiled from city B to the farthest city from it, which happens to be different from A. Prove that, if his exiles continue the same way, he will never return to city A.
9. (MHO 2012) In a galaxy far, far away, there is a United Galactic Senate with 100 Senators. Each Senator has no more than three enemies. Tired of their arguments, the Senators want to split into two parties so that each Senator has no more than one enemy in his own party. Prove that they can do this. (Note: If A is an enemy of B, then B is an enemy of A.)
10. Fifty pairs of lasers on the floor of a lab are being shone at each other. If two laser beams (from A to B and from C to D) are crossing, it is permitted to uncross them by making A and C shine at each other and making B and D shine at each other. Prove that eventually all lasers will become uncrossed.