Functional equations

UW Math Circle – Advanced Group

Session 5 (24 October 2013)

1. (Moscow 1981) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+k)(1-f(x)) = 1 + f(x)$ for all x, for some $k \neq 0$. Show that f is periodic.

2. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies $f(xy) = f(x)+f(y)$ for all x and y. Given that $f(2013) = 1$, find $f(\frac{1}{2013})$.

3. (Canada 1969) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function with the following properties:

   - $f(2) = 2$;
   - $f(mn) = f(m)f(n)$ for all m and n;
   - $f(m) > f(n)$ whenever $m > n$.

   Prove that $f(n) = n$ for all n.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x)+f(1-x) = x^2$ for all x. (Hint: Replace x with $1-x$ to obtain a linear system in the two variables $f(x)$ and $f(1-x)$.)