Metric Topology III

UW Math Circle – Advanced Group

Session 7 (7 November 2013)

A set S in the plane or on the line is *bounded* if there is a disc containing S. It is *compact* if it is closed and bounded.

An infinite sequence of points (a_n) has an *accumulation* point at x if any disc around x contains infinitely many of the points in the sequence. A sequence could have

- No accumulation points $(0, 1, 2, 3, 4, \dots)$. By the Bolzano-Weierstraß theorem – see below – every sequence has an accumulation point if we take the one-point compactification of the plane/line, i.e., add the point at ∞.
- One accumulation point $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- Two accumulation points $(1, 0, 1, 0, 1, 0, \dots)$
- Infinitely many accumulation points (list all the rational numbers in a sequence).

A sequence (a_n) *converges* to x if, for any disc D around x, a_n is eventually in D. Equivalently, there are only finitely many terms of the sequence outside of D.

We proved the following:

- If a sequence converges to x, then x is an accumulation point of the sequences.
- A sequence converges to at most one point.
- (Bolzano-Weierstraß theorem) A set S is compact if and only if every sequence of points in S has a subsequence converging to a point in S.

The following is another definition of compactness. A set of open sets is an *open cover* of a set S if S is in the union of these open sets. Let's call a set S *really compact* if, given any open cover of S, a finite subset of the open sets used in the cover would suffice to cover S. The *Heine-Borel theorem* states that a set is compact if and only if it is really compact.