Problem Set 8

UW Math Circle

Session $\omega + 14$ (29 January 2015)

"Since I'm really three people, I normally experience the average happiness. On a good day, I experience the sum. No one can be happier than another. It's all the same."
– Alexandre Dumas, "The Three Musketeers"

1. Suppose $a, b, c > 0$ and L, M, N are three means. Prove or disprove: if the set $\{a, b, c\}$ is the same as the set $\{L(a, b, c), M(a, b, c), N(a, b, c)\}$, then $a = b = c$. What if the means are strictly monotonic?

2. (Putnam 2012) A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

3. (Belarus 1964) Suppose $x > y > 0$. Show that

$$\frac{(x-y)^2}{8x} \leq A(x, y) - G(x, y) \leq \frac{(x-y)^2}{8y},$$

where G and A denote the geometric and arithmetic means, respectively.