# Problem Set 9

UW Math Circle

Session $\omega + 15$ (5 February 2015)

1. Start with two numbers $a_0$ and $b_0$, where $0 < a_0 \leq b_0$. Let $a_1 = G(a_0, b_0)$ and $b_1 = A(a_0, b_0)$. Continue this process, at each step taking $a_{n+1} = G(a_n, b_n)$ and $b_{n+1} = A(a_n, b_n)$. Note that $0 < a_n \leq b_n$.

   (a) Show that the sequence $a_n$ is (weakly) increasing and $b_n$ is (weakly) decreasing, and that $a_n \leq b_n$.

   (b) Show that $b_n \to a_n$ approaches 0, and conclude that both sequences approach a single number, called the *arithmetic-geometric mean* of $a_0$ and $b_0$.
   (There is a "nice" expression for this number:

   $$\frac{\pi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{\sqrt{a_0^2 \cos^2 t + b_0^2 \sin^2 t}}}.$$

   The integral in the denominator is called an elliptic integral of the first kind and, in general, does not have a closed form.)

   (c) Suppose $M_p$ and $M_q$ are two power means (above, $p = 0$ and $q = 1$). Show that the analogous process will converge unless $p = -\infty$ (minimum mean) and $q = +\infty$ (maximum mean). (At each step we let $a_{n+1} = M_p(a_n, b_n)$, $b_{n+1} = M_q(a_n, b_n)$.) What if we take three numbers and three means?

2. Show that among all $n$-gons inscribed in the unit circle, the regular $n$-gon has the greatest area. What about perimeter? (Hint: Connect every vertex to the center of the circle. It may help to remember that $\sin x$ is a concave function for $0 < x < 180°$ and $\cos x$ is a concave function for $0 < x < 90°$.)

3. Russell and Tom play a game with two piles of stones. The first pile contains 88 stones, and the second pile contains 72. On his turn, a player can remove from a pile a number of stones that is a divisor of the number of stones currently in the pile. (For example, on the first turn, he can remove 1, 2, 11, or all 88 stones from the first pile.) A player who cannot make a move loses. Russell goes first. Who wins?