Problem Set 10

UW Math Circle

Session $\omega + 18$ (26 February 2015)

1. (BAMO-8 2015) Members of a parliament participate in various committees. Each committee consists of at least two people, and it is known that every two committees have at least one member in common. Prove that it is possible to give each member a colored hat (hats are available in black, white or red) so that every committee contains at least two members with different hat colors.

2. (Moscow City 1966) In a certain city, every subway line has at least four stations, at most three of which are transfer stations (i.e. are shared with another line). No three lines meet at a single station. It is known that any station can be reached from any other making no more than two transfers. Determine the greatest possible number of subway lines in the city.

3. (Moscow City 1973) In a certain subway system (undirected graph), it is possible to reach any station from any other station. Prove that it is possible to close one station so that it is possible to reach any remaining station from any other remaining station.

4. The triangular numbers are the numbers $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for $n = 1, 2, 3, \ldots$. Prove that infinitely many triangular numbers are also perfect squares.