

# Problem Set 6

UW Math Circle

Session  $\omega + 8$  (13 November 2014)

1. (a) Show that if you place a map of Seattle on the table, there will be some point on the map that is exactly on top of the point on the ground that it represents (assuming, of course, that you are in Seattle).  
Note that this is still true if you first crumple the map, throw it on the ground, and step on it.  
(b) Show that this may be false if you are permitted to rip / cut the map.
2. Let  $\chi$  be the least number of colours needed to colour any map drawn on the Möbius strip. The Möbius strip is considered to be “transparent” (only has one side).
  - (a) Show that  $\chi \geq 6$  by drawing a map that cannot be coloured with 5 colours.
  - (b) Show that any planar graph has a vertex of degree 5 or less. (Hint: Recall Euler’s formula  $V - E + F = 2$  and the inequality  $3F \leq 2E$ .)
  - (c) Conclude the same for a graph drawn on the Möbius strip.
  - (d) Show by induction that any graph drawn on the Möbius strip can be coloured with 6 colours, so  $\chi \leq 6$ . Combined with (a) this shows  $\chi = 6$ .

This was first shown by Heinrich Tietze in 1892. You can see his original paper (in German) by following the link at the Math Circle website.

