Perfect Numbers, Mersenne Primes, and the Euclid-Euler Theorem

Thomas Browning

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We say N is perfect when the sum of all of the factors of N equals $2N$. For example, the factors of 6 are $1, 2, 3, 6$ and $1 + 2 + 3 + 6 = 2 \cdot 6$. Another example is 28 which has $1, 2, 4, 7, 14, 28$ as its factors and $1 + 2 + 4 + 7 + 14 + 28 = 2 \cdot 28$. The only known perfect numbers are

n	nth Perfect Number	Number of Digits	Year of Discovery
1	6	1	Known to Ancient Greeks
2	28	2	Known to Ancient Greeks
3	496	3	Known to Ancient Greeks
4	8128	4	Known to Ancient Greeks
5	33550336	8	1456
6	8589869056	10	1588
7	137438691328	12	1588
8	2305843008139952128	19	1772
9	265845599...953842176	37	1883
10	191561942...548169216	54	1911
$\vdots$			
49	451129962...930315776	44677235	2016

As you can see, perfect numbers grow exponentially and we are continuing to find them. Although perfect numbers are simple to define, there are some famous unsolved problems about them:
1) Are there infinitely many perfect numbers?
2) Are there any odd perfect numbers?
In order to better understand perfect numbers, we will use the function $\sigma(n)$ which is the sum of the factors of n. The values of $\sigma(n)$ for the first 30 integers are

n	$\sigma(n)$	n	$\sigma(n)$	n	$\sigma(n)$
1	1	11	12	21	32
2	3	12	28	22	36
3	4	13	14	23	24
4	7	14	24	24	60
5	6	15	24	25	31
6	12	16	31	26	42
7	8	17	18	27	40
8	15	18	39	28	56
9	13	19	20	29	30
10	18	20	42	30	72

Notice that perfects number occur precisely when $\sigma(n) = 2n$. Here are some questions to think about:
1) What is $\sigma(p)$ when p is prime number?
2) What is $\sigma\left(2^{k-1}x\right)$ when x is odd?
3) Try to find the prime factorization of the first couple perfect numbers. Is there a pattern?

We now have the conjecture that

N is an even perfect number if and only if $N = 2^{k-1}\left(2^k - 1\right)$ where $2^k - 1$ is a prime number.

This is the well-known Euclid-Euler Theorem. The proof of this theorem is broken up into two parts. First we will show that if $N = 2^{k-1}\left(2^k - 1\right)$ where $2^k - 1$ is prime, then N is an even perfect number. Then we will show that if N is an even perfect number then $N = 2^{k-1}\left(2^k - 1\right)$ where $2^k - 1$ is a prime number. To prove this, we will use the following properties of $\sigma(n)$:

$$\sigma(p) = p + 1 \text{ where } p \text{ is a prime number.} \tag{1}$$

$$\sigma\left(2^{k-1}x\right) = \left(2^k - 1\right)\sigma(x). \tag{2}$$

Now we will suppose that $N = 2^{k-1}\left(2^k - 1\right)$ where $2^k - 1$ is prime and we will try to compute $\sigma(N)$:

$$\sigma(N) = \sigma\left(2^{k-1}\left(2^k - 1\right)\right) = \left(2^k - 1\right)\sigma\left(2^k - 1\right) = \left(2^k - 1\right)\left(2^k - 1 + 1\right) = 2\left(2^k - 1\right)\left(2^{k-1}\right) = 2N$$

which shows that N is an even perfect number. We are halfway done! Now we suppose that N is an even perfect number. Then we will factor out all of the 2's out of N to write N as $2^{k-1}x$ where x is odd. Then

$$\left(2^k - 1\right)\sigma(x) = \sigma\left(2^{k-1}x\right) = 2\left(2^{k-1}x\right) = 2^k x.$$

Then $2^k - 1$ is a factor of $2^k x$. Since $2^k - 1$ is odd, $2^k - 1$ must be a factor of x. Then we can divide $x/\left(2^k - 1\right)$ and call this y. Then if we go back to our equation and divide both sides by $2^k - 1$ we get that

$$\sigma(x) = 2^k y.$$

Notice that x and y are both different divisors of x, so

$$2^k y = \sigma(x) = x + y + (\text{other factors of x}).$$

But $x + y = 2^k y$ so x must have no other factors. Then x has two factors which means that x must be prime and y must be 1. Then $x = 2^k - 1$ and

$$N = 2^{k-1}x = 2^{k-1}\left(2^k - 1\right) \text{ where } 2^k - 1 \text{ is a prime number.}$$

We are now done with the proof of the Euclid-Euler theorem. Here are some more questions to think about:
1) Try to find the value of k for the first couple perfect numbers. Is there a pattern?
2) Show that all even perfect numbers end in a 6 or an 8.
3) Show that all even perfect numbers, with the exception of 6, are 1 mod 9.
4) Show that all even perfect numbers are triangle numbers.