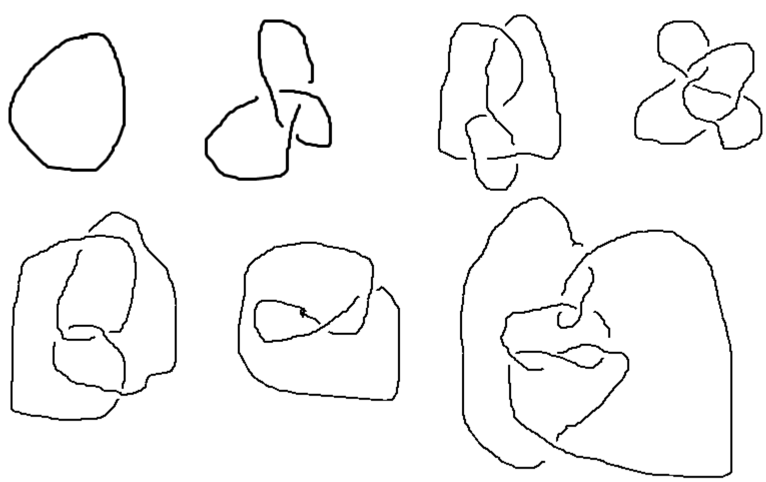
UW Math Circle

May 26th, 2016

We think of a knot (or link) as a piece of string (or multiple pieces of string) that we can stretch and move around in space– we just aren't allowed to cut the string. We draw a knot on piece of paper by arranging it so that there are two strands at every crossing and by indicating which strand is above the other. We say two knots are equivalent if we can arrange them so that they are the same.

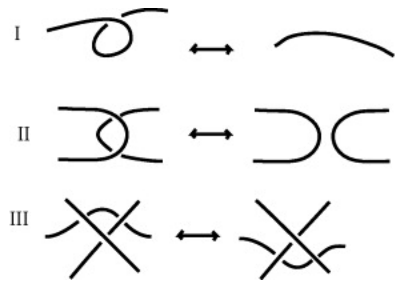
1. Which of these knots do you think are equivalent? Some of these have names: the first is the unkot, the next is the trefoil, and the third is the figure eight knot.



2. Find a way to determine all the knots that have just one crossing when you draw them in the plane. Show that all of them are equivalent to an unknotted circle.

The Reidemeister moves are operations we can do on a diagram of a knot to get a diagram of an equivalent knot. In fact, you can get every equivalent digram by doing Reidemeister moves, and by moving the strands around without changing the crossings.

Here are the Reidemeister moves (we also include the mirror images of these moves).



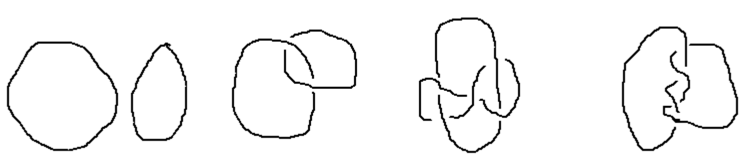
We want to have a way to distinguish knots and links from one another, so we want to extract some information from a digram that doesn't change when we do Reidemeister moves.

Say that a crossing is postively oriented if it you can rotate it so it looks like the left hand picture, and negatively oriented if you can rotate it so it looks like the right hand picture (this depends on the orientation you give the knot/link!)

For a link with two components, define the linking number to be the absolute value of the number of positively oriented crossings between the two different components of the link minus the number of negatively oriented crossings between the two different components, divided by two. In symbols: $\left|\frac{\#\text{positive crossings} - \#\text{negative crossings}}{2}\right|$. **You only count crossings between the different strands of the link!**

1. Show that the linking number doesn't depend on the orientation of the link, and that the linking number doesn't change when you do Reidemeister moves. So, if you have two links with different linking numbers, they must actually be different links!

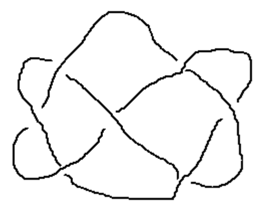
2. Compute the linking numbers of these links. The first is called the unlink, the next the Hopf link, and the third the Whitehead link.



We say that a knot or a link is tricolorable if we can color a diagram of it with three different colors (using at least two of them) so that at each crossing the their strands have either all the same color or all three colors (the color must be the same along each arc of the digram, it can only change at crossings).

1. Show that if a diagram has a tricoloring, then so does the diagram we get after doing any Reidemeister move. So if one knot/link is tricolorable and another isn't, they must be differerent knots/links!

2. Show that this knot isn't equivalent to the unknot.



3. Show that the trefoil isn't equivalent to the figure eight knot.

4. Show that the Whitehead link isn't equivalent to the unlink (remember, these two links have the same linking number).

5. Here is a generaliation of tricoloring. Label the arcs of a digram $0, 1, \ldots, n-1$, and say that we have an n-coloring if at each crossing, the sum of the arcs going under is equal to twice the arc going over, modulo n.

   (a) Show that n-colorability is a knot invariant (it doesn't change when we do Reidemeister moves).

   (b) Reinterpret tricolorability in terms of n-colorabilty.

   (c) Show that the figure eight knot is not equivalent to the unknot.

6. Take a digram of the figure eight knot, and switch the positions of the strands at each crossing. Is the resulting knot equivalent to the one you started with? What happens if you do this with the trefoil?