UW Math Circle

March 17, 2016

1. (a) You have an equilateral triangle in the plane, and you label the vertices $1, 2, 3$. You can move the triangle by rotating it and by flipping it over, but after you are finished it must fit back exactly where it started, but there may be a different ordering on the vertices. How many permuations of one, two, and three can you get by doing this? Write your answers in cycle notation. Do you get all of them?

   (b) You do the same thing, except now you start with a square. What permutations of one, two, three, four can you get? Do you get all of them?

   (c) Instead of allowing any movement of the square, let's say the only allowed operations on the square are the 90 degree clockwise rotation (P) and the reflection around the vertical axis (Q). If we can do these operations as many times as we want, and in whatever order we want (so, for example, we could do $QPPPQP$), what permutations can we get? Do we get the same answer as in part (b)?

   (d) What about with a regular n-gon? Using the cycle notation for permutations, find a compact way to write the possible permutations you can get. (You may find it helpful to try it for a few other shapes, like a regular pentagon or a hexagon. If this seems hard, can you instead figure out how many permutations you get?)

2. (a) You have a cube. You can move it around however you like, but you must place it back so it fits exactly where it started. How many different ways can you put it back? (Hint: can you think of this problem in terms of permutations?)

   (b) If you are also allowed to reflect the vertices of the cube over internal axes (like switching the front and back face of the cube), how many different ways can you move the cube?

   (c) The number you got in part (a) should be the same as the total number of permutations of $1, 2, 3, ..., n$ for some n. What is that n? Can you find exactly n things that get permuted during the movements of the cube?

3. (a) Given a permutation of $1, 2, 3, ..., n$ written in cycle notation, what is the probability that the permutation has a cycle of length $n - 1$? What about $n - 2$?

   (b) Given a permutation of $1, 2, 3, ..., 2n$ written in cycle notation, what is the probability that the permutation has a cycle of length n?

Finished early? Try to solve this problem!

The director of a prison offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategybut may not communicate once the first prisoner enters to look in the drawers. Find a strategy that has a success rate of at least 30%.