UW Math Circle

December 7, 2017

1. Here are the first 15 rows of Pascals's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

Choose 5 colors corresponding to the numbers $0, 1, 2, 3$, and 4. Color each number with the color corresponding to its remainder modulo 5. Compare this to the pictures on your homework!

2. Again, here are the first 15 rows of Pascals's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

Now, do the same thing with remainders modulo 4. Compare this to the triangle you colored modulo 2 on your homework!

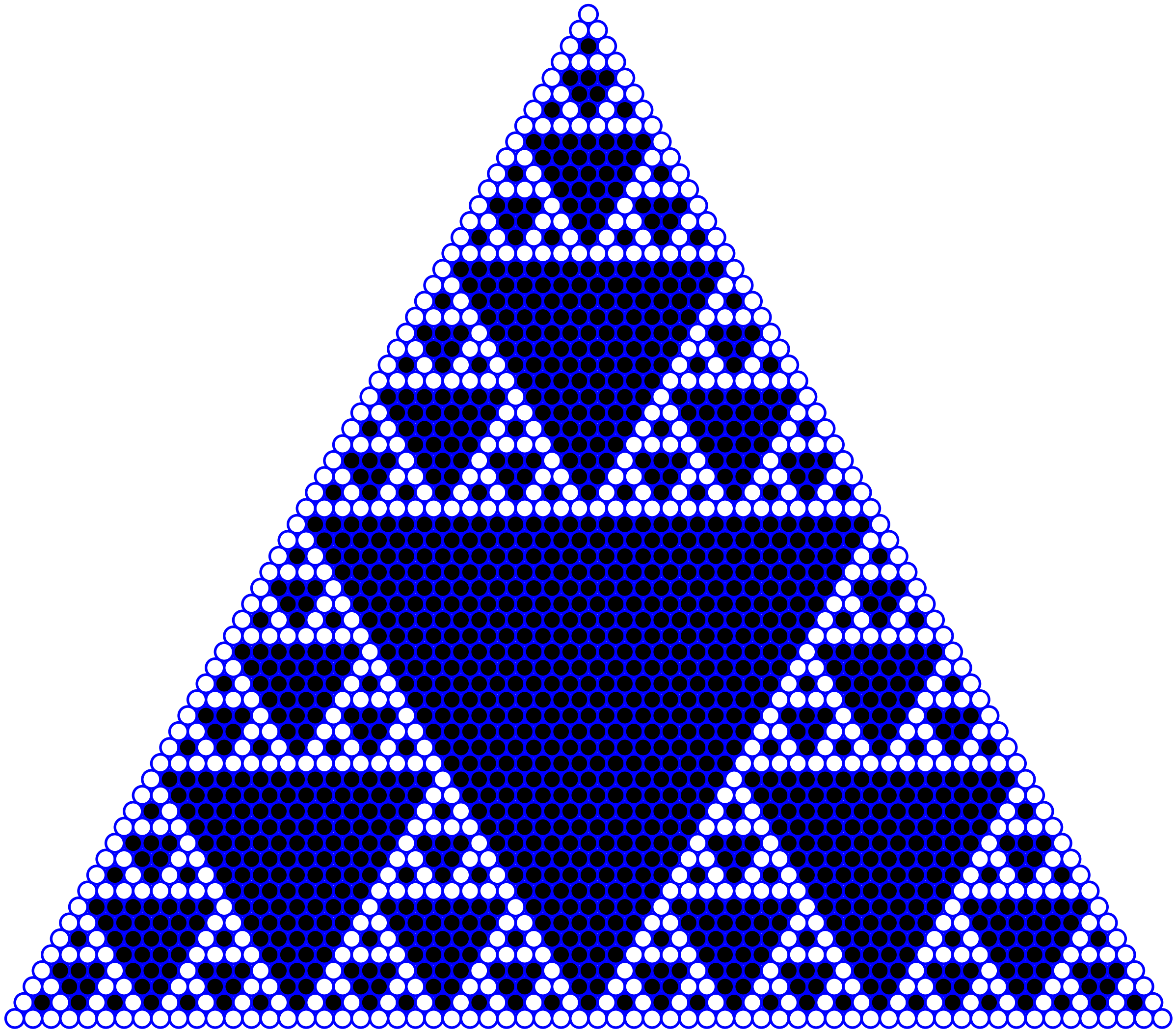
Your goal is to find and explain patterns in the given pictures of the triangles, colored by remainder $\mod p$ for various numbers p. Each number is colored white if it not is divisible by p and black if it is divisible by p. Here are some questions you can think about:

1. Are there any rows that are all (or mostly) $0 \mod p$? What about rows that are all not 0? Why or why not?
2. How often do rows that are mostly $0 \mod p$ appear?
3. Can you explain the "upside down" triangles that appear?
4. How do you think the triangle would look if we colored it by remainders $\mod 6$? Would it relate to the $\mod 2$ and $\mod 3$ triangles?

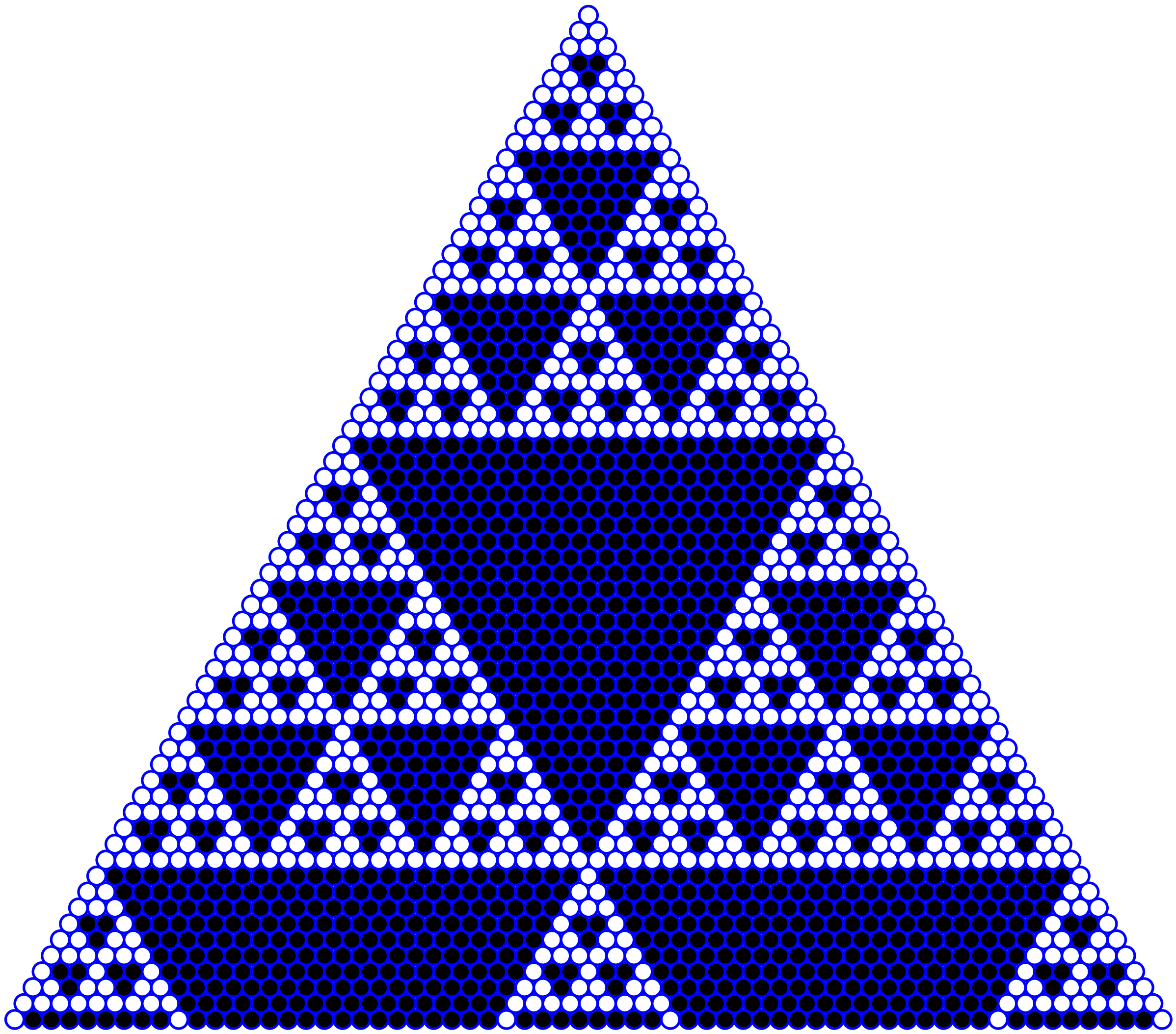
Find as many patterns as you can!

You can find many more patterns in (the uncolored) Pascal's Triangle itself. Find *and explain* as many patterns as you can!

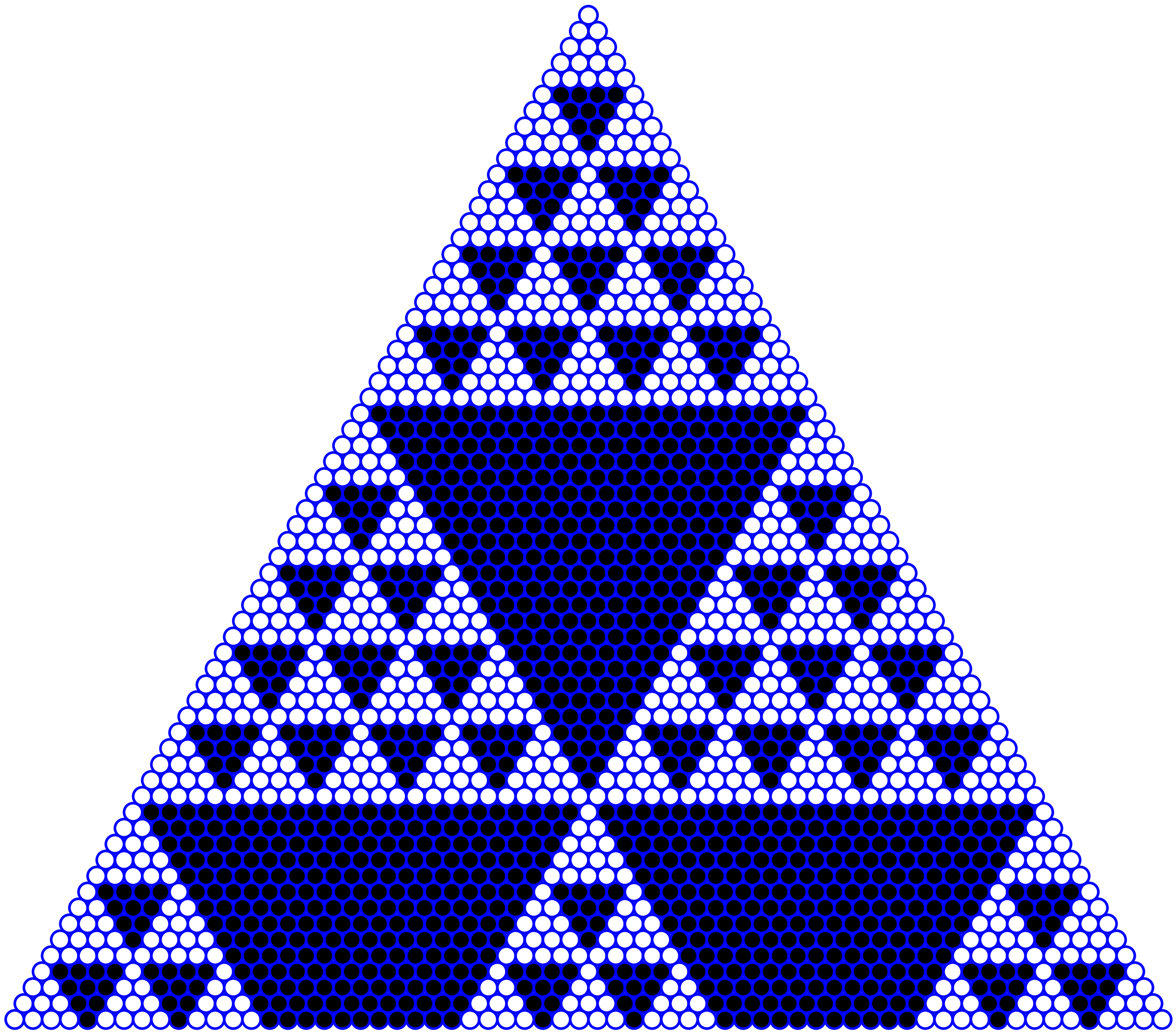
Coloring mod 2:



Coloring mod 3:



Coloring mod 5:



We've been talking about finding remainders in the numbers $\binom{n}{m}$ (the numbers that appear in Pascal's Triangle). Our goal is to find a way to compute these in an easier way.

Let's talk remainders when you divide by 3. We'll talk about remainders and compare them to something from their base 3 expansion. For example, let's consider the number $\binom{14}{10}$.

1. Actually compute $\binom{14}{10}$. What is it? What is the remainder when you divide by 3?

2. We can write 14 and 10 in base 3 (using the exploding dots machine exploding 3 dots and turning it into 1 dot!). We know

$$14 = 1 \times 9 + 1 \times 3 + 2 \times 1$$

$$10 = 1 \times 9 + 0 \times 3 + 1 \times 1$$

   Matching the corresponding coefficients, we could compute

$$\text{coefficients of 9: } \binom{1}{1} = 1$$

$$\text{coefficients of 3: } \binom{1}{0} = 1$$

$$\text{coefficients of 1: } \binom{2}{1} = 2$$

   What is the product of these three numbers? How is this related to your answer in part 1?

3. Let's try again!

   (a) What is $\binom{14}{9}$? What is the remainder when you divide by 3?

   (b) Write 14 and 9 in their base 3 expansions, i.e. as $? \times 9 + ? \times 3 + ? \times 1$.

$$14 = \_\_\_ \times 9 + \_\_\_ \times 3 + \_\_\_ \times 1$$

$$9 = \_\_\_ \times 9 + \_\_\_ \times 3 + \_\_\_ \times 1$$

(c) Matching the corresponding coefficients, compute the binomial coefficients (like in problem 2)

coefficients of 9:

coefficients of 3:

coefficients of 1:

What is their product? How does it compare to your answer in (a)?

4. Pick at least 3 other binomial coefficients and try the same thing.

Here are some extra problems about counting!

1. What is the number of ways to place 2 rooks on an 8×8 chessboard so that they don't attack one another? (A rook attacks all of the squares in its row or column.)

2. From a deck of 52 cards, how many different five-card poker hands could be dealt?

3. If Steve has 10 coins, how many ways are there for him to arrange them in a sequence with exactly 5 heads if the first coin must be a head? What if the first coin cannot be a head?

4. Sarah has one week's worth of dog treats: 2 bones, 2 biscuits, and 3 pieces of jerky. She gives her dog one treat each day. How many ways does Sarah have to distribute her treats?

5. Samuel the squirrel is collecting nuts for winter. He store the nuts in three different trees. If he has collected 10 nuts so far, how many ways are there for him to distribute them among the three trees? (He doesn't have to put nuts in all of the trees, for example he could put all in the first tree and none in the others.

6. How many ways are there to give 11 pieces of candy to 5 students, if the first student must get at least 2 pieces of candy?

7. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?