UW Math Circle

October 12, 2017

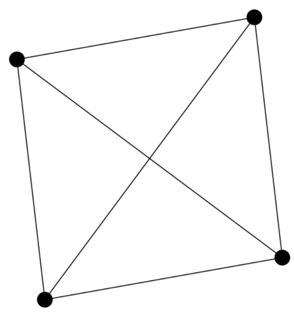
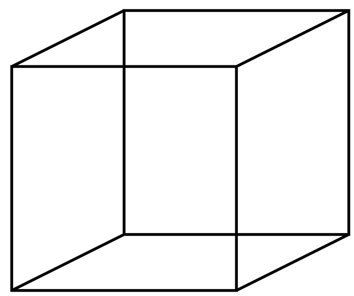
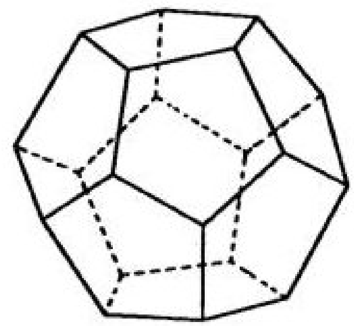
1. (a) There are three houses, and each house needs to be connected to the water tower and also to the chocolate milk tower. Is it possible to run pipes from each house to the two towers so that the pipes don't intersect? You can move the houses and towers around as you like.

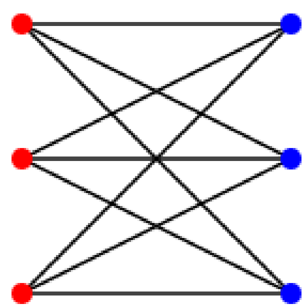
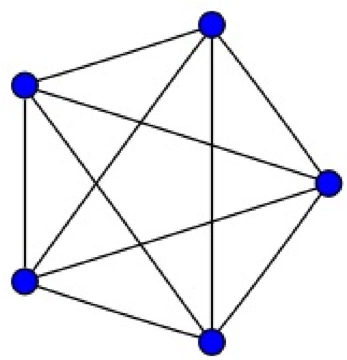
   (b) Now there are four houses, and each house wants to install a gondola connecting them to the other three houses. Can they do this without the gondolas intersecting?

   (c) There are again three houses, and each house wants to connect to the water, chocolate milk, and cherry coke towers. Can they do this without the pipes intersecting?

2. In each of the following drawings, the dots represent islands and the edges represent desired bridges between the islands. For the cube and the one that looks a bit like a soccer ball, the islands are at the corners. The city wants to build these bridges in a way so no two bridges cross.

   Which configurations of islands below can have the bridges built in a way so that the bridges don't cross?

3. For each configuration of islands where the bridges don't cross, count the number of islands (I), bridges (B), and the number of different bodies of water (W) the bridges divide the ocean into (note that the water outside all of the bridges counts as 1!). Do you notice anything? In each case, what is $I - B + W$?

4. For the other configurations, can you still count I, B, and W? Do any of these numbers change if you build a bridge in a different way?

We call the configurations of islands with bridges that don't cross ***planar graphs****. Remember from last time that graphs are made of vertices (islands) and edges (bridges). We don't allow multiple edges between the same vertices or edges from a vertex to itself.*

5. If V is the number of vertices, E the number of edges, and F the number of faces (the number of regions the edges divide the plane including the region outside of the graph), what are V, E, and F in terms of I, B, and W?

6. Show that, for any planar graph with at least two edges, $2E \geq 3F$.

7. Show that, for any planar graph, $V - E + F = 2$.

8. Use those two equations to show that, for a planar graph, $E \leq 3V - 6$. Can you use this to show that some of the graphs on the first page aren't planar? Are there any graphs that satisfy this inequality but you think aren't planar?

9. Show that a planar graph with at least 3 vertices and no cycle of length three satisfies the inequality $2E \geq 4F$. (A cycle is a path from a vertex back to itself.) Use it to show that, in this case, $E \leq 2V - 4$. Does this help you to decide if some of the graphs on the first page aren't planar?

10. Can you find a graph that satisfies the inequality $E \leq 2V - 4$ but isn't planar?

11. Show that a planar graph with at least three vertices has a vertex with five or fewer edges coming out of it.