Let's introduce some different symbols, so we can talk about all groups at once! Instead of saying "consider the group of integers" or "think about the symmetries of a pentagon", we'll say "consider the group $G$": this means that "$G$" is a placeholder that might mean any group you like.

When $G$ is an arbitrary group like this, we usually use the letter "$e$" to mean the identity element, and letters like "$g$", "$h$", "$a$", "$b$" and "$c$" to mean arbitrary elements. Instead of saying "combine $g$ and $h$", we can write "$g * h$": the "$*$" is another placeholder, that might mean "$+$" or "$\times$" or "...then do this mattress flip..." or whatever the group combining operation is.

In this language, the definition of a group is:

- a set $G$ and an operation $*$, so that
- there's an identity element $e$ in $G$ such that $e * g = g$ and $g * e = g$ for any element $g$ in $G$,
- every $g$ in $G$ has an inverse element, written $g^{-1}$, so that $g * g^{-1} = e$ and $g^{-1} * g = e$, and
- the operation $*$ is associative, which we can finally explain: it means that $(a * b) * c = a * (b * c)$ for any $a$, $b$ and $c$ (where the parentheses mean "do the thing inside the parentheses first").[1]

**Problem 0.** A group $G$ can only ever have one identity element. Why?

**Problem 1.** Every element $g$ in $G$ has only one inverse. Why?

**Problem 2.** Suppose that $g * h = e$. Explain why $h * g$ must also equal $e$, so $h$ must be $g^{-1}$.

[1] Associativity means we can write things like "$g_1 * g_2 * g_3 * g_4 * \cdots$" without parentheses, since it doesn't matter what order we do the $*$'s in. Can you think of an operation that isn't associative?

**Problem 3.** Sometimes the order of group elements matters, and sometimes it doesn't. Explain why the following statement is true: $a * b$ and $b * a$ are equal if and only if $a * b * a^{-1} * b^{-1} = e$.

**Problem 4.** If $g$ is any element of $G$, the set

$$\langle g \rangle = \{e,\ g,\ g*g,\ g*g*g,\ g*g*g*g,\ \ldots,\\ g^{-1},\ g^{-1}*g^{-1},\ g^{-1}*g^{-1}*g^{-1},\ \ldots\}$$

is a subgroup of $G$.

Remember, to check if a set $S$ is a subgroup of $G$, you need to check:

- If you multiply two things in $S$, you get something else in $S$.
- The identity element is in $S$.
- Everything in $S$ has an inverse in $S$.

**Problem 5.** Assume that $G$ is a finite set. Explain why $g * g * \cdots * g = g^{-1}$ for some number of $g$'s.

**Problem 6.** Suppose $H$ is a subgroup of $G$ and $g$ is any element of $G$. Let $gHg^{-1}$ be the set of things you can write as $g * h * g^{-1}$ for some $h$ in $H$. Explain why $gHg^{-1}$ is a subgroup of $G$.

**Problem 7.** Remember that a group is called "cyclic" if it can be generated by one element. Explain why $a * b = b * a$ for every $a$ and $b$ in a cyclic group.

**Problem 8.** Explain why every finite group can be found as a subgroup of the group of permutations of $n$ things, for some $n$.