# Probability, primes, and mods

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Let's recall some basic probability. Suppose N is a uniformly chosen random integer between 1 and 100: this means that every integer has equal probability of being chosen,

$$\mathbb{P}(N=1) = \mathbb{P}(N=2) = \mathbb{P}(N=3) = \cdots = \mathbb{P}(N=100) = \frac{1}{100}.$$

The expected or average value of N is

$$\mathbb{E}N = 1 \cdot \mathbb{P}(N = 1) + 2 \cdot (N = 2) + \dots + 100 \cdot \mathbb{P}(N = 100)$$
  
=  $\frac{100 \cdot 101}{2 \cdot 100}$   
= 50.5

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- No! If all the probabilities were equal, they would all have to be 0!
- For today, when we pick a 'random' integer, it means: pick a uniformly random integer between 1 and 100, or 1 and 1000, or...

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Discuss!

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**Interpretation:** Divisibility by distinct primes are **independent events**. For distinct primes p and q,

$$\mathbb{P}(p \text{ and } q \text{ divide } N) = \mathbb{P}(p|N) \cdot \mathbb{P}(q|N)$$

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This has to happen for every prime! Using the independence for different primes,

$$\mathbb{P}(N,M ext{ relatively prime}) pprox \left(1-rac{1}{2^2}
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This product can be written in a nicer form. First let's re-write all the terms: for any p,

$$1 - \frac{1}{p^2} = \left(\frac{1}{1 - 1/p^2}\right)^{-1} = \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \frac{1}{p^6} + \cdots\right)^{-1}$$

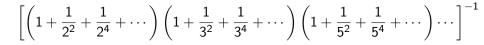
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So the product becomes

$$\left[\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\cdots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\cdots\right)\cdots\right]^{-1}$$



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Think about which terms appear in this sum when you multiply it out: any number n that can be written as a product of squares of prime numbers appears exactly once in the denominator. But those are just the square numbers! So

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$$\mathbb{P}(N, M ext{ relatively prime}) = \left[1 + rac{1}{2^2} + rac{1}{3^2} + rac{1}{4^2} + rac{1}{5^2} + \cdots 
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Think about which terms appear in this sum when you multiply it out: any number n that can be written as a product of squares of prime numbers appears exactly once in the denominator. But those are just the square numbers! So

$$\mathbb{P}(N, M \text{ relatively prime}) = \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right]^{-1} = \frac{6}{\pi^2}.$$

## So this says that if N and M are uniform between 1 and $10^6$ ,

$$\mathbb{P}(\mathsf{gcd}(\textit{N},\textit{M})=1)pprox 6/\pi^2pprox .601$$

#### N and M share no common factors with probability around 3/5.

**Question:** For any prime *p*, what's the biggest power of *p* that divides *N*?

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More in the exercises...

**Question:** What is the probability that N is prime?

It turns out that  $\mathbb{P}(N \text{ is prime}) \approx 0(!)$  More precisely:

Theorem (Prime number theorem)

Let  $\pi(x)$  denote the number of primes less than or equal to x. Then

$$\pi(x) \approx \frac{x}{\log x}$$

Thus, if N was chosen between 1 and  $10^6$ , then

$$\mathbb{P}(\mathsf{N} ext{ is prime}) pprox rac{1}{\log 10^6} pprox rac{1}{14}$$

The primes behave sort of 'uniformly random'. For example, the four possible last digits in base 10 of the primes are 1, 3, 7, and 9, and:

### Theorem (Dirichlet, 1837)

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This works in any base.

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Theorem (Vinogradov, 1937)

All sufficiently large odd numbers can be written as a sum of three primes.

For many purposes, we can pretend that every number x was chosen to be prime with probability  $\frac{1}{\log x}$ .

God may not play dice with the universe, but something strange is going on with the prime numbers. - Paul Erdős

# Coverings

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- Example 1: 0 mod 2 and 1 mod 2 is a covering, since every number is either even or odd. The score is 1/2 + 1/2 = 1.
- Example 2: 0 mod 2, 1 mod 3, 3 mod 6, 5 mod 6 is a covering, since all the congruence classes mod 6 appear. The score is 1/2 + 1/3 + 1/6 + 1/6 = 7/6.

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Your job: find coverings with **MINIMUM** possible score, and such that you don't use the LCM of your mods as one of the mods. (Neither of the above examples passes this test, since 2 is the LCM in example 1, and 6 is the LCM in example 2.)

(Bonus: can you construct coverings with arbitrarily big scores?)