

UW Math Circle, Winter 2020

modern arm bun

This week we talked about conditional probabilities and expectations. Two cool facts we used:

For two random variables X and Y , the expected value of $X + Y$ is

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

And for two events A and B , the conditional probability of A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

Some problems:

Suppose I flip a fair coin 5 times.

1. Assume that heads comes up exactly three times out of five. What is the probability that the first flip was heads? What's the probability that the last flip was heads? What's the probability that the first two flips were heads? Check by writing down all the possibilities (how many are there?), and then also by using the conditional probability formula.
2. What is the expected number of times the sequence HTH appears? For example, it appears once in the sequence TTHTH and twice in the sequence HTHTH. What if the coin was flipped 100 times?
3. What's the probability that the sequence HTH *doesn't* appear? What if the coin was flipped 100 times?
4. Suppose that the first and last flips had the same result: either both heads or both tails. What is the probability that all five of the flips were heads? Is this different from the 'usual' probability (not conditioned on this event) of getting the sequence HHHHH?

Some non-coin-flipping problems:

5. I have 10 lengths of string. I pick one end of string at a time and connect it to another end, chosen randomly from all the other remaining ends, until there are no more ends. What is the expected number of closed loops I end up with? For example, if I connect each string to itself, I could have 10 loops; or I could possibly connect the strings into a single giant loop. (Hint: think about the 'string connecting process' recursively.)
6. Suppose I form a 'random' permutation on the numbers $1, 2, \dots, 5$ by sending 1 to a random number between 1 and 5, then sending 2 to a different number between 1 and 5, and so on – so I end up with some permutation, say 12435. What is the expected number of values that get sent to themselves? For example, in the permutation 12435, 1, 2, and 5 are all sent to themselves. What is the probability that exactly one element is sent to itself? Exactly two elements are sent to themselves?
7. I have a bag with four balls: two are red, one is blue and one is green. I draw two balls simultaneously from the bag. Suppose one of the drawn balls is red. What is the probability that the other one is red? Blue? Green?
8. Suppose I roll three dice, and get three numbers X, Y , and Z . What is the probability that there is a triangle with side lengths X, Y, Z ?