HOMEWORK #2, DUE 10/14

Math 504A

1. Show that the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ of two copies of $\mathbb{Z}_2$ is isomorphic to the infinite dihedral group, that is the semidirect product of $\mathbb{Z}$and $\mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ acts on $\mathbb{Z}$ by sending any integer to its negative.

2. Show that the group $P = PSL_2(\mathbb{Z})$ is isomorphic to the free product of $\mathbb{Z}_3$ and $\mathbb{Z}_2$, as outlined below.

(a) Show that the matrices $A' = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate the group $S = SL_2(\mathbb{Z})$, by first showing that the matrix $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is generated by A' and B'. Then, given any matrix $M \in S$, show how to multiply M on the left by products of suitable powers of C and B' to perform any desired row operation on it (preserving the determinant as 1) with integer coefficients. Using the Euclidean algorithm, transform the first column of M into $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by such operations, and then observe that M must now be a power of C.

(b) It follows that the images A, B in P generate P; note that A has order 3 while B has order 2. Now show that no nonempty product of elements in P that are alternately A or A^2 and B can equal 1. (Look at the linear fractional transformations T_1, T_2, T_3, corresponding to A, A^2, B, respectively, and observe that T_1 maps positive irrational numbers to negative irrational numbers less than -1, T_2 maps positive irrational numbers to negative irrationals greater than -1, and finally that T_3 sends negative to positive irrationals.) Deduce the desired result.

3. Find products X_1, X_2 of A, B corresponding to the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ in P and show that the X_i freely generate a subgroup of P (which turns out to have finite index).

4. Classify the subgroups of index two of the free group F_2 on two generators x, y, giving a set of free generators of each such subgroup.

5. Find a subgroup of F_2 that is free on infinitely many generators and give the generators explicitly.