Final Exam–Math 505

1. Classify the ideals in a discrete valuation ring.
The unique maximal ideal in such a ring is generated by a single element x; then the nonzero ideals are the principal ones (x^i) generated by a power of x (including $i = 0$).

2. The ring $\mathbb{Z}[\sqrt{-11}]$ is not a Dedekind domain. How can one modify this ring slightly to make it a Dedekind domain?
Replace the ring by its integral closure $\mathbb{Z}[(1+\sqrt{-11})/2]$.

3. Show that the product $\mathbb{P}^1 \times \mathbb{P}^1$ is not homeomorphic to $\mathbb{P}^2$, by studying the behavior of curves in these two spaces. (Here $\mathbb{P}^1 \times \mathbb{P}^1$ has the Zariski topology, not the product topology.)
Any two curves in $\mathbb{P}^2$ intersect, by a result in class; but this is not so for $\mathbb{P}^1 \times \mathbb{P}^1$: if say x_1, x_2 are projective coordinates for the first copy of $\mathbb{P}^1$, then the curves defined by $x_1 = 1, x_2 = 1$ and $x_1 = 1, x_2 = 2$ have empty intersection.

4. Let S be the splitting field of $x^4 - 2$ over $\mathbb{Q}$, a Galois extension. Work out the intermediate fields between $\mathbb{Q}$ and S of degree 2 over $\mathbb{Q}$ explicitly and indicate to which subgroups of the Galois group of S they correspond.
The fields are $\mathbb{Q}[\sqrt{2}], \mathbb{Q}[\sqrt{-2}]$, and $\mathbb{Q}[i]$, corresponding to the subgroups $< r >, < r^2, s >$ $, < r^2, rs.$ of the Galois group D_4 of S over $\mathbb{Q}, D_4$ the dihedral group of order 8, generated by a rotation r of order 4 and a reflection s of order 2.

5. Let L be the splitting field of the polynomial $x^3 - 3$ over a field K. Give the conditions on K for L to be a Galois extension of it and determine all the possibilities for the Galois group of L over K whenever L is Galois over K.
L is always Galois over K; the only possibly characteristic for K is 3, but in that case the polynomial reduces to x^3, so that $L = K$ is still Galois over K. The possible Galois groups are 1 (if $x^3 - 3$ already splits over K), $\mathbb{Z}_2$ (if $x^3 - 3$ has a root in K but 1 had not primitive cube root), $\mathbb{Z}_3$ (if the reverse situation applies), or S_3 (if no cube root of 3 or primitive cube root of 1 is present in K).

6. Give the conditions on a Dedekind domain A for it to admit a finitely generated nonfree projective module M.
A should have nontrivial class group, or equivalently *not* be a PID.

7. Compute the integral closure of the coordinate ring A of the subvariety of $\mathbb{C}^2$ with equation $x^4 = y^7$ (that is, the set of elements in the quotient field of A that are integral over it). What variety has this integral closure as its coordinate ring?
Identifying the coordinate ring $K[A] \cong K[x,y]/(x^4 - y^7)$ of A with $K[t^7, t^4]$, the integral closure is $K[t]$, the coordinate ring of the line K^1.

8. Compute the strict transform (or blowup) of the variety with defining equation $y^3 = x^3 + x^4$ at the singular point $(0,0)$.
The equations defining the transform are $y = ux, y^3 = x^3 + x^4$, or equivalently $u^3 = x + 1$, so this blowup identifies with $\{(u^3 - 1, u(u^3 - 1), y); u \in K\}$, or just K itself, K the basefield.