HW #6, DUE 2-22

1. Let C be the ring of continuous real-valued functions on the unit interval $[0,1]$ with addition, subtraction, multiplication and division defined pointwise. Use the compactness of $[0,1]$ to show that the weak Nullstellensatz holds for C: every proper ideal I is such that its variety $V(I) = \{x \in [0,1] : f(x) = 0, f \in I\}$ is nonempty. (On the other hand, the strong Nullstellensatz fails for C: it admits nonmaximal ideals I whose varieties reduce to a single point.)

2. Show that the ring $C^\infty(\mathbb{R})$ of infinitely differentiable real valued functions on $\mathbb{R}$ is not Noetherian,. (Use the functions f_a defined for $a \in \mathbb{R}$ by $f_a(x) = e^{-1/(x-a)^2}$ if $x \neq a$ while $f_a(a) = 0$; you may assume that all derivatives of f_a are 0 at a).

3. For K an algebraically closed field let $V \subset K^3$ be the curve $\{(t^3, t^4, t^5) : t \in K\}$. Show that V is an irreducible affine subvariety of K^3 whose ideal I is generated by the three elements $xz - y^2, x^2y - z^2, x^3 - yz$ of $K[x,y,z]$. Identify the coordinate ring $K[V]$ as a subring of $K[t]$. Show that I is not generated by any two elements of $K[x,y,z]$, by looking at the degrees of the terms of its elements.

4. Show that any affine variety in K^n is compact in the Zariski topology.

5. Compute the intersection in $\mathbb{P}^3$ of the varieties defined by $x^2 - yw = 0$ and $xy - zw = 0$ and show that this intersection is reducible. Identify its components.