LECTURE 4-21

We now show that for well-behaved algebras over local rings, it suffices to test for flatness at the maximal ideal only. The Local Criterion for Flatness states that *if R is a local Noetherian ring with maximal ideal M and S a local Noetherian R-algebra with maximal ideal N such that $MS \subset N$, and if P is a finitely generated S-module, then P is flat over R if and only if* $\mathrm{Tor}_1(R/M, P) = 0$.

To prove this we note first that the condition is necessary be a standard characterization of flatness. Suppose that it is satisfied. We first show that $\mathrm{Tor}_1(P', P) = 0$ for any R-module P' of finite length, by induction on the length. Indeed, if P' has length 1, so is irreducible, then $P \cong R/M$ since P is generated by any of its nonzero elements, and the result follows by hypothesis; in general, we set up a short exact sequence with middle term P' and outer terms of smaller length than P' and apply the inductive hypothesis together with the long exact sequence in Tor. Now let I be any ideal of R and let $u \in I \otimes_R P$ lie in the kernel of the multiplication map from $I \otimes_R P$ to P We must show that $u = 0$. Now $I \otimes_R P$ has an S-module structure, inherited from the one on P, and the hypothesis guarantees that $M^n(I \otimes_R P) \subset N^n(I \otimes_R P)$. Then $I \otimes_R P$ is finitely generated and S is local, so the Krull intersection theorem guarantees that $\cap_n N^n(I \otimes_R P) = \cap_n (M^n(I \otimes_R P) = 0$. Hence it is enough to show that $u \in M^n(I \otimes_R P$ for all n. The module $M^n(I \otimes_R P)$ is the image in $I \otimes_R P$ of $M^n I \otimes_R P$, and the Artin-Rees Lemma implies that $M^t \cap I$ lies in $M^n I$ for sufficiently large t, so it is enough to show that u lies in the image of $(M^t \cap I) \otimes_R P$ for all t. Tensoring the short exact sequence $0 \to M^t \cap I \to I \to I/(M^t \cap I) \to 0$ with P produces the exact sequence $(M^t \cap I) \otimes_R P \to I \otimes_R P \to I/(M^t \cap I) \otimes_R P \to 0$, so it suffices to show that u goes to 0 in $I/(M^t \cap I) \otimes_R P$. Tensoring the homomorphism $I \to I/(M^t \cap I)$ with P and using the multiplication map to get an induced homomorphism from P to $R/M^t \otimes_R P$, we note that the image of u in P is 0, so it suffices to show that the kernel of the map $I/(M^t \cap I) \otimes_R P \to R/M^t \otimes_R P$ induced from the inclusion of I into R is trivial. Identifying $I/(M^t \cap I)$ with $(I + M^t)/M^t$, we see that this last map is also induced from the inclusion of $I + M^t$ into R. But now $\mathrm{Tor}_1(R/(I + M^t), P) = 0$ since $R/(I + M^t)$ is annihilated by M^t and so has finite length as an R-module. An appeal to the long exact sequence in Tor concludes the proof.

If R and R' are any rings, $R \to R'$ is a homomorphism, and M is a flat R-module, then $R' \otimes_R M$ is flat as an R'-module because tensoring over R' with $R' \otimes_R M$ is the same as tensoring over R with M. Now a simple lemma computes the Tor groups of M/xM in terms of the Tor groups of M, for any $x \in R$ that is not a zero-divisor on R or M. More precisely, it states that *$Tor_i i^{R/(x)}(N, M/xM) \cong Tor_i^R(N, M)$ if M is an R-module, x an element of R that is not a zero-divisor on R or M, and N is an $R/(x)$-module.* To see this we tensor a free resolution of M over R with $R/(x)$ to get a chain complex of free $R/(x)$-modules whose last term maps only M/xM. The homology of this complex is $\mathrm{Tor}^R(R/(x), M)$ which under our hypotheses on x and M is 0 in all positive degrees, so the chain complex is in fact a free resolution of M/xM. Tensoring with N and taking homology, we get the desired formula.

Combining the lemma with the Local Criterion, we see that *if R is a local Noetherian ring with maximal ideal M, S a local Noetherian R-algebra whose maximal ideal N contains S, P a finitely generated S-module, and finally if $x \in M$ is a non-zero-divisor on R and M, then M is flat over R if and only if M/xM is flat over $R/(x)$.* Indeed, if M is flat over R, then $M/xM = R/(x) \otimes_r M$ is flat over $R/(x)$, as we remarked above. Conversely, if M/xM is flat over $R/(x)$ then $\mathrm{Tor}_1(k, M/xM) = 0$. By the above lemma, $\mathrm{Tor}_1(k, M) = 0$, so M is flat over R by the Local Criterion.

The geometric interpretation of this corollary is the following. Let X, Y be affine varieties over a field k and suppose that we have morphisms $\phi : X \to Y, \psi : Y \to k$. We call these morphisms flat if the corresponding algebra homomorphisms $k[t] \to k[Y] \to k[X]$ make $k[X]$ resp. $k[Y]$ flat as modules over $k[Y]$ resp. $k[t]$; flatness of $k[X]$ and $k[Y]$ over $k[t]$ will usually hold because $k[t]$ is a PID. For each $p \in k$ we have a map from the fiber Y_p of Y over p (the inverse image of p in Y and the fiber X_p over p in X (the inverse image of p under the composite map $\phi\psi$). Then the corollary says that if the map $X_p \to Y_p$ is flat in a neighborhood of p" in X_p, then the map $X \to Y$ is flat in a neighborhood of p" in X, so if the fibers of $X_p \to Y_p$ behave nicely near a point, then the fibers of $X \to Y$ behave nicely near the same point.