LECTURE 4-24

We return to Atiyah-Macdonald for this lecture, covering material from parts of Chapters 1,2,3, and 5, mostly in the exercises. This material proves results for general rings which are proved in Eisenbud only for Noetherian rings (indeed, Eisenbud has a tendency to run too quickly to the Noetherian hypothesis). First, let A, B be any rings and $f : A \to B$ a homomorphisms. Given an ideal I of A its *extension* I^e is the ideal generated by $f(I)$ in B; similarly, given an ideal J of B its *contraction* J^c is the inverse image $f^{-1}(J)$, an ideal in A. If J is prime, so is J^c, but there is no corresponding result for I^e; also J^c need not be maximal even if J is. Clearly $I \subset I^{ec}, J \supset J^{ce}$. If C, E denote the set of contracted ideals in A resp. extended ideals in B (i.e. those of the from J^c resp. I^e) then the map $I \to I^e$ is a bijection from C onto E whose inverse sends J to J^c. In the special case $B = A_S$, the localization of A by the multiplicatively closed subset S, then every ideal of B (as previously observed) is extended; if I is an ideal of A then $I^{ec} = \cup_{s\in S}(I : s) = \cup_{s\in S}\{x : xs \in I\}$. The map $I \to I^e$ defines an order-preserving bijection from the prime ideals of A not meeting S to the prime ideals of A_S. If $A \to B$ is a ring homomorphism and P is a prime ideal of A, then P is the contraction of a prime ideal of B if and only if $P^{ec} = P$, for if $P^{ec} = P$ and S is the complement of P in A, then P^e does not meet S, whence its extension in B_S is proper and contained in a maximal ideal M. Then the contraction M^c of M in B is prime, contains P^{ec} and thus P, but does not meet S, so it must contract to P, as desired.

We now return to the notion of flatness. We have already seen that an A-module M is 0 if and only if its localization M_P is 0, or even its localization M_N at every maximal ideal N is 0. It follows that an A-module map $M \to N$ is injective if and only if the localized map $M_P \to N_P$ is injective for all prime ideals P, or just for maximal ideals. As a consequence M is flat over A if and only if M_P is flat over A_P for all prime ideals P, or (again) just for maximal ideals. We need one more simple fact about tensor products. Given a ring homomorphism $A \to B$ and a B-module N, we may regard N as an A-module via the homomorphism and so can form the tensor product $N_B = B \otimes_A N$ (more generally, we use this notation for any A-module N). We have the multiplication map $m : N_B \to N$ sending $b \otimes y$ to by. Its restriction to $1 \otimes N$ is then injective since the composition $N \to 1 \otimes N \to N$ is the identity. Hence N_B is the direct sum of the image $1 \otimes N$ of N in it and the kernel of m.

Now we can relate flatness to extension and contraction of ideals. Let B be a flat A-algebra. Then the following are equivalent: (1) $I^{ec} = I$ for all ideals I of A; (2) the induced map Spec $B \to$ Spec A is surjective; (3) for every maximal ideal M of A its extension M^e is not all of B; (4) for every nonzero A-module M the module M_B is not 0; (5) for every A-module M the map $x \to 1 \otimes x$ from M to M_B in injective. Indeed, (1) implies (2) by the fact about contracted prime ideals observed in the first paragraph; applying (2) to maximal ideals we get (3); given (3), if N is a nonzero A-module and $x \in N, x \neq 0$, then by flatness it is enough to show that $N'_B \neq 0, N' = Ax$; but N' takes the form A/I for some ideal I of A, whence $N'_B \cong B/IB \neq 0$ since I lies in a maximal ideal M and $MB \neq B$. If (4) holds, let M' be the kernel of $M \to M_B$. Since B is flat over A the sequence $0 \to M'_B \to M_B \to (M_B)_B$ is exact. But we saw in the previous paragraph that the map from M_B to $(M_B)_B$ in injective, so $M'_B = 0$. Finally, if (5) holds and I is an ideal of A, then $(A/I)_B \cong B/IB$ contains A/I as a submodule, whence $I^{ec} = I$, as desired. We say that B is *faithfully flat* over A if these conditions hold. Any flat homomorphism $A \to B$ (realizing B as a flat A-module) becomes faithfully flat upon suitable localization: if Q is prime in B with contraction $Q^c = P$ in A, then B_P is flat over A_P by above remarks and B_Q is a localization of B_P so it too is flat over A_P (localization is exact); moreover, the extension of the unique maximal ideal PA_P in B_Q is no bigger than QB_Q, so B_Q is faithfully flat over A_P; in particular, the map Spec $B_Q \to$ Spec A_P is surjective.

As an example, let $A = \mathbb{Z}$ and let B be the Gaussian integers $\mathbb{Z}[i]$. Then B is free as a Z-module and so flat over A and clearly $pB \neq B$ for any prime p in A, so B is faithfully flat over A. The map Spec $B \to$ Spec A is surjective, but not every prime ideal in B is extended, for if $p \in A$ is prime and congruent to 1 mod 4, then it is well known that two primes in B contract to (p), each generated by a Gaussian integer of norm p, but neither generated by p itself.