LECTURE 5-17

We begin by proving the lemma stated last time, that a torsion-free module M over a one-dimensional Noetherian domain R satisfies $\ell(M/xM) \leq r(M)\ell(R/(x))$ for any $x \in R, x \neq 0$, with equality if M is finitely generated. Suppose first that M is finitely generated. Let $m_1, \ldots, m_r$ be a K-basis of $K \otimes_R M$ (with K the quotient field of R); multiplying each m_i by a suitable nonzero element of R, we may assume that the m_i lie in M. Now we have a copy of R^r inside M, generated by the m_i; let N be the quotient of M by this copy of R^r. Recalling the long exact sequence in Tor and the earlier calculation that $\mathrm{Tor}_1(P, R/xR) = P^x = \{p \in P : xp = 0\}$ for any R-module P, we get an exact sequence

$$0 \to N^x \to R^r/xR^r \to M/xM \to N/xN \to 0$$

and another sequence

$$0 \to N^x \to N \to N \to N/xN \to 0$$

where the third map is multiplication by x. The finite generation of M and N imply that there is a nonzero $f \in R$ with $fN = 0$ (since the m_i are a K-basis for $K \otimes M$), so N and N^x have finite length, being finitely generated modules over $R/(f)$. Taking the alternating sum of lengths, we get $\ell(N^x) = \ell(N/xN), \ell(M/xM) = \ell(R^r/xR^r) + \ell(N/xN) - \ell(N^x) = \ell(R^r/xR^r) = r\ell(R/xR)$, proving the lemma in this case. In general, if the lemma fails for some infinitely generated module, necessarily of finite rank r, then it would also have to fail for some finitely generated submodule of at most this rank, contradicting the previous argument; so it holds in general.

The case $M = K$ shows that equality can fail if M is not finitely generated. Now we close out Chapter 11 by showing that *the integral closure of a Noetherian domain R of dimension one is again Noetherian and of dimension at most one; in fact, any ring S lying between R and a finite extension L of its quotient field K is Noetherian of dimension at most one and has only finitely many ideals containing a given nonzero ideal of R. In particular, the integral closure of R in L is again Noetherian (though it need not be finitely generated as an R-module)*; this is called the *Krull-Akizuki Theorem*. To prove this, let J be a nonzero ideal of S; as the elements of S are algebraic over R, one sees immediately that J contains a nonzero element a of R. Now all assertions follow at once if we can show that J/aS has finite length, and for this it is certainly enough to show that S/aS has finite length as an R-module. Since $K \otimes S \subset L, S$ is a torsion-free R-module of finite rank, so we are done by the above lemma.

The situation worsens rapidly if the hypotheses on R are weakened. Indeed, we have already noted that for R as above, the subring S need not be finitely generated as an R-module (though it will be if L is a finite separable extension of K, by a Galois-theoretic argument using the trace given last quarter). If R has dimension 2, then Mori and Nagata have shown that the integral closure of R is Noetherian, but in this case there may be subrings lying between R and this closure that are not Noetherian. If R has dimension at least 3, then the integral closure of R need not be Noetherian.

Turning now to Chapter 13 in Eisenbud, we return to the class of commutative rings of greatest relevance for algebraic geometry, namely *affine rings*, or finitely generated k-algebras with k a field (not necessarily algebraically closed). Along the lines of what we last discussed, we begin with a famous result of Emmy Noether, asserting that *the integral closure of an affine domain R in any finite extension L of its quotient field K is finitely generated as an R-module, and so in particular is again an affine domain.* To prove this, we recall first by Noether normalization (the key step in our proof of the Nullstellensatz) that any such R is itself a finitely generated integral extension of a polynomial ring $k[x_1, \ldots, x_d]$, where d is the dimension of R, so we may assume that $R = k[x_1, \ldots, x_d]$. Now it does no harm to replace L by its normal closure, so that L/K is a normal extension. Finally, letting L' be the fixed field of the Galois group of L over K, we find that L' is a purely inseparable extension of K, meaning that either $L = K$ or else K has characteristic $p > 0$ and every element x of L' is such that $x^q \in K$ for some power $q = p^k$ of p. We will first show that the integral closure R' of R in L' is finitely generated over R. Since L' is finite over K, there is in fact a single power q of p such that L' is generated over K by qth roots of rational functions. Extending k to a new field k' by adjoining the qth roots of their coefficients, we may assume that $L' = k'(x_1^{1/q}, \ldots, x_d^{1/q})$. Then the integral closure of R in L' is $T = k'[x_1^{1/q}, \ldots, x_d^{1/q}]$, since this ring is integrally closed, has quotient field L', and is finite over R. Since $R' \subset T$, this shows that R' is finite over R. Now we invoke the result proved last quarter that the integral closure of a Noetherian domain S in a finite Galois extension of its quotient field is finitely generated as an S-module to deduce the desired result (you can read a different proof of the last step in Proposition 13.14 on p. 298 (2004 edition) of Eisenbud).