LECTURE 5-19

A remarkable consequence of the finiteness of the integral closure for affine domains (proved last time) is the following: *Let k be an algebraically closed field of characteristic 0. The algebraic closure of the field $k((x))$ of Laurent series over k is the union over all positive integers n of the fields $k((x^{1/n}))$, and the integral closure of $k[[x]]$ in $k((x^{1/n}))$ is $k[[x^{1/n}]]$.*. Indeed, if L is a finite extension of $k((x))$, then it suffices to show that the integral closure T of $k[[x]]$ in L takes the form $k[[x^{1/n}]]$ for some n, so that $L = k((x^{1/n}))$. We know that T is finite over $k[[x]]$. It follows from a simple corollary (7.6) to a consequence of Hensel's Lemma (p. 190, §7.3 of the 2004 edition of Eisenbud) that T is a direct product of complete local domains. Since T is itself a domain, it must be complete and local, and of dimension one since it is an integral extension of a ring of dimension one. Hence T is a DVR. Write π for a generator of its maximal ideal. For some n we must have $x = u\pi^n$, with u a unit of T. The residue field (quotient by the maximal ideal) of T is finite over k, hence equal to k since k is algebraically closed; it also follows that the image $\overline{u}$ of u in $T/(\pi)$ has an nth root $\overline{v}$. Since the characteristic of k is 0, the polynomial $t^n - \overline{u}$ has a simple root $\overline{v}$, so Hensel's Lemma implies that $\overline{u}$ lifts to an nth root v of u in T. Letting $\pi' = v\pi$, we see that π' is an nth root of x in T. The map $k[[x']] \to T$ sending x' to $v\pi$ is then onto by Theorem 7.16 in Eisenbud (proved earlier in class) and must be an isomorphism since $\dim T = 1$, so the result follows. As an immediate corollary we get that *any polynomial equation $f(x, y) = 0$ in two variables over an algebraically closed field of characteristic 0 admits solutions of the form $y = p(x^{1/n})$ for some Laurent series p, which may be taken to be a power series if f is monic in y. If in addition $f(0, 0) = 0$, then y may be written as a power series in $x^{1/n}$ without constant term.* Indeed, the irreducible factors of $f(x, y) = 0$ over $k((x))$ must have roots y in some finite extension $k((x^{1/n}))$. If f is monic in y then these roots are integral and lie in $k[[x^{1/n}]]$. If in addition $f(0, 0) = 0$ then at least one of the roots must reduce mod x to 0 and thus lie in the maximal ideal of $k[[x^{1/n}]]$.

The two most striking properties of affine domains not shared by general commutative rings are firstly that their dimensions are also given by the transcendence degrees (maximum number of algebraically independent elements over the basefield k) of their quotient fields (this follows from Noether normalization), and secondly that they are *universally catenary* in the sense that given any prime ideals P, Q with $P \subset Q$, then any two maximal chains of prime ideals from P to Q (i.e. chains such that one cannot insert any prime properly between two consecutive elements) have the same length (we proved this last quarter as part of our discussion of dimension of algebraic varieties). It follows at once that *if $R \subset T$ is an inclusion of affine domains over a field k and K is the quotient field of R, then* $\dim T = \dim R + \dim K \otimes_R T$, since the dimension of R equals the transcendence degree of K over k, the dimension of $K \otimes_R T$ equals the transcendence degree of $K \otimes_R T$ over K, and whenever we have a chain of fields $K_0 \subset K_1 \subset K_2$, the transcendence degree of K_2 over K_0 is the sum of the transcendence degrees of K_1 over K_0 and K_2 over K_1. A ring-theoretic consequence is a version of *Nagata's altitude formula*, stating that *if R is a Noetherian domain, T a finitely generated R-algebra that is also a domain, Q a prime ideal* *of T contracting to $P \subset R$, then* $\dim T_Q \leq \dim R_P + \dim K \otimes_R T$, *where K is the quotient field of R; if R is universally catenary and Q is maximal among primes contracting to P, then equality holds*. For the proof see Theorem 13.8, p. 292, in Eisenbud. We also get a refinement of Noether normalization for affine rings: *if R is an affine ring of dimension d over a field k and if $I_1 \subset \cdots \subset I_m$ is a chain of ideals with* $\dim I_j = d_j, d_1 > d_2 > \cdots > d_m \geq 0$ *(recall that the dimension of an ideal I is by definition the codimension of the quotient ring R/I), then R contains a polynomial ring $S = k[x_1, \ldots, x_d]$ in such a way that R is a finitely generated S-module and $I_j \cap S = (x_{d_j+1}, \ldots, x_d)$ for $j = 1 \ldots, m$. If the I_i are homogeneous, then the x_i may be taken to be homogeneous. If k is infinite and R is generated over k by $y_1, \ldots, y_r$, then for $j \leq d_m$ the element x_j may be chosen to be a k-linear combination of the y_i* (Theorem 13.3 in Eisenbud). In geometric terms, given a d-dimensional variety $X \subset k^m$ and a chain of subvarieties of X, there is a finite map from X to the affine space k^d such that the subvarieties are mapped to coordinate planes.