VIRTUAL LECTURE 5-29

Given any ring homomorphism $f : A \to B$ there is an induced map f^* taking Spec(B) to Spec(A), which is continuous in the Zariski topologies on these spectra; this map can be used in conjunction with localization and passage to the quotient to relate the spectrum of a difficult ring to that of an easier one. As an example, we will simultaneously work out the spectra of $\mathbb{Z}[x]$ and $k[x, y], k$ a field; these two very different rings exhibit many parallel features. Letting B be one of these rings and $A = \mathbb{Z}$ in the first case, $A = k[x]$ in the second, we begin by observing that any prime ideal of B contracts to a prime ideal in A; if this is nonzero, then by passage to the quotient we are reduced to studying prime ideals of $\mathbb{Z}/(p)[x]$ in the first case (for p a prime integer) and prime ideals of $k'[x]$ in the second (where k' is a simple finite extension of k, obtained as the quotient of $k[x]/(q), q$ an irreducible monic polynomial over k. In both cases the ring obtained is a PID, so we deduce immediately that *prime ideals of* $\mathbb{Z}[x]$ *meeting* $\mathbb{Z}$ *nontrivially take the form* (p, f), *where* p *is a prime integer and* $f \in \mathbb{Z}[x]$ *is a monic polynomial whose reduction* $\overline{f}$ *mod* p *is irreducible in* $\mathbb{Z}/(p)[x]$. Similarly, *prime ideals in* $k[x, y]$ *meeting* $k[x]$ *nontrivially take the form* (q, q'), *where* q *is monic irreducible in* $k[x]$ *and* q' *is monic in* y, *its reduction* $\overline{q}'$ *mod* x *being monic irreducible in* $k[y]$. It remains to treat prime ideals of $Z[x]$ contracting to 0 in $\mathbb{Z}$, and similarly prime ideals of $k[x, y]$ contracting to 0 in $k[x]$; here localization comes to the fore, telling us that such ideals are in bijection to prime ideals of $\mathbb{Q}[x], k(x)[y]$, respectively. As both of these rings are again PIDs, we deduce that *the remaining nonzero prime ideals in* $\mathbb{Z}[x]$ *take the form* (f) *for* $f \in \mathbb{Z}[x]$ *irreducible* (equivalently, $f \in \mathbb{Z}[x], f$ irreducible as a polynomial in $\mathbb{Q}[x]$, and f primitive). Once again a parallel result holds for $k[x, y]$.

More generally, homomorphisms $A \to B$ behave especially nicely on the level of prime spectra if B is flat as an A-module. Recall from the fall quarter that this means (by definition) that tensoring with B is an exact functor from A-modules to A-modules, and that B is automatically flat over A whenever B is projective (in particular free) as an A-module. Things are even nicer if B is *faithfully flat*; here (by one of several equivalent characterizations) the induced map from Spec(B) to Spec(A) is surjective, (equivalently, the extension of any maximal ideal of A is proper in B). There is however no requirement that the map from Spec(B) to Spec(A) be injective, and indeed B can have a much larger spectrum than A in this situation.

Apart from flat extensions, the other kinds of especially nice ring extensions from this point of view are localizations. The localization $B = A_S$ of a ring A by a multiplicatively closed subset S is such that every ideal of B is the extension of an ideal of A. Though in general different ideals of A can have the same extension in B, this cannot happen for prime ideals: *there is a one-to-one inclusion-preserving correspondence between prime ideals of* B *and prime ideals of* A *not meeting* S, *given by extension from* A *to* B *and contraction from* B *to* A. (The same holds for primary ideals and was used to derive the uniqueness results in class for primary decompositions.) This phenomenon helps explain the term "localization"; in the special case where S is the complement of a fixed prime ideal P, localization cuts out all prime ideals except those "Zariski-close" to P in the sense that they are contained in it. (In a similar but more elementary manner, moding out by P cuts out all primes except those *containing* P.) If S is the complement of P, then in particular A_S is a local ring, whose unique maximal ideal is generated by P.

Recall also from last quarter that integral extensions of a ring also behave nicely in this respect. We saw then (in our current language) that given an integral extension $A \subset B$, every prime ideal of A is the contraction of some prime ideal of B; although there can be two primes in B with the same contraction in A, there cannot be any inclusion relations between two such ideals. In particular, the (Krull) dimensions of A and B always coincide.