LECTURE 5-3

We now return to Eisenbud, starting with Chapter 10. We have already proved most of the results in this chapter last term, but there are a few new ones and some new terminology to introduce. The main theme of the section in the book including this chapter is dimension theory; we begin by recalling two of the three equivalent definitions of the dimension of a Noetherian ring R. The first is the Krull dimension, namely the supremum of the lengths n of all strictly increasing chains $P_0 \subset \ldots \subset P_n$ of prime ideals in R. If we consider only chains of prime ideals ending at a fixed one P, then we have called the maximum length of any such chain the height of P; following Eisenbud, we now rechristen it the *codimension of P*. The *dimension* $\dim P$ of P is now defined to be the dimension of the quotient ring R/P. Likewise the dimension $\dim M$ of an R-module M is defined to be the dimension of $R/I, I$ the annihilator of M. The second definition of dimension applies to a local ring R with maximal ideal M and is the least number of generators of any ideal containing a power of M. Using this equivalence we showed that any prime ideal P of a Noetherian ring R that is minimal over a principal ideal (x) has codimension at most 1; the codimension equals 1 if and only if x is not a zero divisor. Using this fact we can characterize UFDs among Noetherian domains. First note that a Noetherian domain R is a UFD if and only if every prime ideal minimal over a principal ideal is itself principal. Indeed, if R is a UFD and $f \in R$ is neither 0 nor a unit, then we can write $f = uf_1 \ldots f_n$ with u a unit and the f_i irreducible. Any prime ideal containing f then contains f_i for some i; conversely unique factorization guarantees that (f_i) is prime for all i. Conversely, if R satisfies this condition on prime ideals and $f \in R$ is neither 0 nor a unit, then f must have an irreducible divisor f_1, lest we get an infinite strictly ascending chain of principal ideals containing (f) in R. Writing $f = f_1 f'$, we similarly find that f', if not a unit, must have an irreducible factor f_2, and so on; then the ascending chain condition forces the sequence $f_1, f_2, \ldots$ to terminate after finitely many terms, with f equal to the product of the f_i times a unit. If P is prime and minimal over (f) then we must have $P = (p)$ for some $p \in R$, forcing p to divide and thus equal f_i for some i. Continuing, we see that f is a product of irreducible prime elements, whence the standard argument (e.g. for $\mathbb{Z}$) shows that R is a UFD. Now since a prime ideal of codimension 1 in a Noetherian domain is automatically minimal over the principal ideal generated by any of its elements, we see that *R is a UFD if and only if every codimension 1 prime ideal is principal.*

More generally, we showed last quarter that any prime minimal over an ideal I generated by c elements has codimension at most c, and conversely any prime of codimension c is minimal over some ideal generated by c elements. Now we can put flatness into the picture: *if R is local with maximal ideal M and if $R \subset S$ is an extension of R with S local and its maximal ideal N containing M, then $\dim S \le \dim R + \dim S/MS$ with equality if S is flat over R.* Indeed, if $d - \dim R, e = \dim S/MS$, then we know that there are $x_1, \ldots, x_d \in M, y_1 \ldots, y_e \in S$ such that $(x_1, \ldots, x_d) \supset M^s, MS + (y_1 \ldots, y_e) \supset N^t$ for some s, t, whence

$$N^{st} \subset (MS + (y_1, \ldots, y_e))^s \subset M^s S + (y_1, \ldots, y_e) \subset (x_1, \ldots, x_d, y_1 \ldots y_e)S$$

and $\dim S \leq d+e$ by the generalized principal ideal theorem. If S is flat as an R-module, then let Q be a minimal prime in S over MS; then $\dim S$ is at least the sum of dimension and codimension of Q. By the going-down property of flat extensions, the codimension of Q is least the dimension of R, so the result follows.

We showed last quarter that the dimension of $k[x_1, \ldots, x_n]$ is n for any field k. Now we can show more generally that $\dim R[x] = \dim R + 1$ for any Noetherian ring R. Given a chain $P_0 \subset \ldots \subset P_n$ of prime ideals in R, one easily sees that each $P_i[x]$ (polynomials in x with coefficients in P_i) is prime in $R[x]$, so we get a chain of primes $P_0[x] \subset \ldots \subset P_n[x] \subset (P_n[x], x)$ in $R[x]$, showing that $\dim R[x] \geq \dim R + 1$. On the other hand, if P has codimension m in R, then P is minimal over $I = (a_1, \ldots, a_m)$ for some $a_i \in R$, whence $P[x]$ is minimal over $I[x]$ (by an easy argument) and $P[x]$ has codimension at most that of P. Now, in homework last quarter, you showed for any ring A that $\dim A[x] \leq 2 \dim A + 1$; in the course of doing this you showed that at most two prime ideals in $A[x]$ could have the same contraction in A. Hence any chain of primes in $R[x]$ can have at most one instance where two consecutive primes have the same contraction in R and $\dim R[x] \leq \dim R + 1$, as desired.