LECTURE 6-2

We conclude with a brief review of completions of rings. Given a ring R and ideal I, its I-adic completion $\hat{R}$ is defined to consist of all sequences $r_1, r_2, \ldots$ with $r_i \in R/I^i, r_i \equiv r_j \mod I^i$ for $i < j$; it is easy to see that the congruence condition makes sense for any $i < j$. Addition, subtraction, and multiplication are all defined componentwise and are easily seen to be compatible with the congruence condition. Roughly speaking, $\hat{R}$ is the smallest ring containing R (each $r \in R$ being identified with the constant sequence $r, r, \ldots$) for which any series $\sum r_j$ with $r_j \in I^j$ converges. In particular, $1 + i$ is unit in $\hat{R}$ for any $i \in I$; its inverse is $1 - i + i^2 - \ldots$. If $I = P$ is maximal, then $\hat{R}$ is local with maximal ideal generated by P. Similarly one completes any R-module M with respect to I by letting $\hat{M}$ consist of all sequences $m_1, m_2 \ldots$ with $m_i \in M/I^iM$ and $m_i \equiv m_j \mod I^iM$ for $i < j$; then $\hat{M}$ is a $\hat{R}$-module in a natural way and the functor sending to M to $\hat{M}$ is exact on finitely generated R-modules if R is Noetherian. The main motivation for completion is that it enables us to solve many equations unsolvable without it; for example, if M is an ideal, R is M-adically complete, and $f(x) \in R[x]$ is a polynomial having $a \in R$ as an approximate root in the sense that $f(a) \equiv 0 \mod (f'(a)^2M)$, then there is a genuine root b of $f(x)$ near a in the sense that $f(b) = 0, b \equiv a \mod (f'(a)M)$; if $f'(a)$ is a non-zero-divisor in R, then b is uniquely determined by these properties. In particular, taking $R = \mathbb{Z}, M = (p)$ for p prime, so that $\hat{R} = \mathbb{Z}_p$, the previously described ring of p-adic integers, then any polynomial $f(x0 \in \mathbb{Z}[x]$ having a *simple* root in the integers $\mathbb{Z}/(p)$ mod p has an honest root in the p-adic integers $\mathbb{Z}_p$; you saw already the necessity for assuming that the root in $\mathbb{Z}/(p)$ is simple in a homework problem last quarter. On the other hand, $\hat{R}$ is not too much bigger than R, being Noetherian whenever R is, having the same dimension as R, and having the associated graded ring $G_{\hat{M}}(\hat{R})$ of $\hat{R}$ with respect to the ideal $\hat{M}$ naturally isomorphic to the corresponding associated graded ring $G_M(R)$ of R with respect to M.

NOW LET'S PARTY!!!