Lecture 10-4: Projective space

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Having mentioned non-affine varieties last time, I will proceed at once to the most important class of such varieties for us, namely projective varieties.

Definitions 1.7.1, p. 14

Projective space $\mathbb{P}^n$ is the set of lines through the origin in $\mathbf{k}^{n+1}$; equivalently, it is the affine space $\mathbf{k}^{n+1}$ with the origin removed, modulo the equivalence relation identifying a nonzero vector x with any nonzero multiple of itself. We write x^* for the equivalence class of x; if $x = (x_0, \ldots, x_n)$ then we call the x_i the homogeneous coordinates of x.

Definitions, p. 14

For $0 \leq i \leq n$ we set $U_i = \{(x_0, \ldots, x_n)^* \in \mathbf{P}^n : x_i \neq 0\}$; we declare the U_i to be open subsets of $\mathbf{P}^n$ and define a bijection $\phi_i : U_i \to \mathbf{A}^n$ via $\phi_i(x_0, \ldots, x_n)^* = (x_i^{-1}x_0, \ldots, x_i^{-1}x_{i-1}, x_i^{-1}x_{i+1}, \ldots, x_i^{-1}x_n)$. We transport the structure of affine variety of $\mathbf{A}^n$ to U_i via this bijection and declare a general subset of $\mathbf{P}^n$ to be open if and only if its intersection with each U_i is open.

A function f defined in a neighborhood of $x \in \mathbf{P}^n$ is defined to be regular if it is regular in the usual sense on some U_i containing x. Thus $\mathbf{P}^n$ has the structure of a prevariety, and in fact that of a variety (check this)

Definition 1.7.1, p. 15

A projective variety is a closed subset of some $\mathbf{P}^n$; a quasi-projective variety is an open subvariety of a projective one.

There is a sharp contrast in the behavior of regular functions in the affine and projective cases: the only regular functions defined at all points of a projective variety are constant!

Now let $I \subset \mathbf{k}[x_0, \ldots, x_n]$ be a homogeneous ideal, so that by definition a polynomial belongs to I if and only if all of its homogeneous terms (of any degree) do. It is easy to check that a nonzero $x \in \mathbf{k}^{n+1}$ is a common zero of all $p \in I$ if and only if any nonzero multiple of x is, so that the set $V(I)$ of common zeros of all polynomials in I may be viewed as a subset of $\mathbf{P}^n$. In fact all nonempty closed subsets of $\mathbf{P}^n$ take the form $V(I)$ for some proper homogeneous ideal I.

The projective version of the Nullstellensatz states that radical homogeneous ideals I of $S = \mathbf{k}[x_0, \ldots, x_n]$ *other than* $J = (x_0, \ldots, x_n)$ are in bijection to closed subsets of $\mathbf{P}^n$, via the map sending I to $V(I)$. In this bijection prime ideals correspond to irreducible closed subsets. Since the ideal J does not occur in this correspondence, it is sometimes called the *irrelevant ideal*.

The product of two projective varieties turns out to have the structure of a projective variety. To see this, one first embeds the product $\mathbf{P}^n \times \mathbf{P}^m$ into $\mathbf{P}^{mn+n+m}$ (not $\mathbf{P}^{m+n}$) via the map sending the pair $(x_0, \ldots, x_n), (y_0, \ldots, y_m)$ of homogeneous coordinates to $(x_0 y_0, \ldots, x_n y_0, x_1 y_0, \ldots, x_n y_m)$. Thus $\mathbf{P}^n \times \mathbf{P}^m$ is *not* isomorphic to $\mathbf{P}^{m+n}$ but rather another projective variety of dimension $m + n$. Then if V, W are two projective subvarieties of $\mathbf{P}^n, \mathbf{P}^m$, respectively, the product $V \times W$ has a natural structure of projective subvariety of $\mathbf{P}^{nm+n+m}$ (Exercise 1.7.5 (4b), p. 16)

The Zariski topology on $\mathbf{A}^n$ makes it into a compact space, but one that behaves rather differently than compact subsets of say $\mathbb{R}^n$ (with the Euclidean topology); for example, any open subset of $\mathbf{A}^n$ is also compact. For this reason I have used the term "quasicompact" rather than "compact". I will show later that projective space, or more generally any projective variety, satisfies a property called *completeness* that makes it look more like compact subsets of $\mathbb{R}^n$ than affine varieties do. For now I just remark that projective varieties are not as central to this course as they are to Math 567; it will be a while before I have occasion to use them.

I conclude by saying a few more words about elliptic curves, which are the most common examples of non-affine algebraic groups. Assume that the characteristic of $\mathbf{k}$ is not 2 or 3 and let $x^3 + ax + b$ be a polynomial in $\mathbf{k}[x]$ with no multiple roots. Consider first the affine subvariety of $\mathbf{k}^2$ defined by the equation $y^2 = x^3 + ax + b$. This is not quite large enough to have a group structure but it acquires one if the extra points at infinity gotten by passing to projective space are added to it.

Accordingly, we homogenize the defining equation, rewriting it as $y^2z = x^3 + axz^2 + bz^3$ and viewing it as an equation in $\mathbf{P}^2$. Then it turns out that given any two points $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in \mathbf{P}^2$ lying in the proejctive variety E defined by this equation there is a unique third point $z = (z_0, z_1, z_2)$ also satisfying it such that x, y, z are collinear in the sense that there are nonzero constants c_0, c_1, c_2 with $c_0x + c_1y + c_2z = 0$. Declaring that $x + y + z = 0$ in this situation makes E into an abelian group, called an *abelian variety*. Its identity element is $(0, 1, 0)$ and the inverse of (x, y, z) is $(x, -y, z)$.