Lecture 10-9: Linearity of algebraic groups and the Jordan decomposition

October 9, 2023

Continuing from last time, I will use the left action of a group on itself to show that every linear algebraic group is a closed subgroup of some GL_n (thus justifying the adjective "linear"). I then introduce the Jordan decomposition, a basic tool from linear algebra which turns out to be very useful in the study of algebraic groups.

Let G be a group and X a G-space. I first prove some basic results about G-orbits on X.

Lemma 2.3.3, p. 28

Any orbit $G.x$ in X is open in its closure. Any orbit of minimal dimension is closed; in particular, closed orbits exist in X.

Proof.

The morphism $a : G \times X \to X$ given by the action restricts to a morphism from any orbit closure of G to X, whence by earlier arguments the orbit is open in its closure. The complement of an orbit in its closure is a union of orbits, which as proper subvarieties of the closure must have smaller dimension. Thus any orbit of minimal dimension is closed. $\square$

Retain the above notation, so that G is a group and X is a G-space, with a the action map.

Proposition 2.3.6, p. 29

Any finite-dimensional subspace V of $\mathbf{k}[X]$ is contained in another finite-dimensional subspace W stable under the action of G. V itself is stable under the action of G if and only if $a^* V \subset \mathbf{k}[G] \otimes V$.

Proof.

It is enough to treat the case where $V = \mathbf{k}f$ is one-dimensional. Write $a^*f = \sum_{i=1}^n u_i \otimes f_i$ with $u_i \in \mathbf{k}[G], f_i \in \mathbf{k}[X]$. For $g \in G$ we have $g.f(x) = \sum_{i=1}^n u_i(g^{-1})f_i(x)$, so that all $g.f$ lie in the subspace W' spanned by the f_i. The subspace W of W' spanned by the $g.f$ is then G-stable, as required; similarly, we see that if $a^*V \subset \mathbf{k}[G] \otimes V$ then V is G-stable. Conversely, if V is G-stable, then let (f_i) be a basis of V and extend it to a basis $(f_i) \cup (g_j)$ of $\mathbf{k}[X]$. Writing $a^*f = \sum u_i \otimes f_i + \sum v_j g_j$, with the $u_i, v_j \in \mathbf{k}[G]$, we have $g.f = \sum u_i(g^{-1})f_i + \sum v_j(g^{-1}g_j$ and all v_j must vanish on G, as desired. $\square$

Now G acts by left translation on itself and this action is easily seen to be faithful (the only element of G acing trivially in this way is the identity). Using this action we have

Theorem 2.3.7, p. 30

There is an isomorphism of G onto a closed subgroup of some $GL_n(\mathbf{k})$.

Proof.

We know that $\mathbf{k}[G]$ is finitely generated as a $\mathbf{k}$-algebra and that a finite set of generators may be enlarged to a basis of a finite-dimensional G-stable subspace V of $\mathbf{k}[G]$, say of dimension n. Any $g \in G$ acting trivially on V also acts trivially on all of $\mathbf{k}[G]$, whence we must have $g = e$. The corresponding homomorphism of G into $GL_n(\mathbf{k}]$ is thus one-to-one and G is isomorphic to a subgroup of $GL_n(\mathbf{k})$. □

The same proof shows that if X is affine and G acts on it, then there is an isomorphism from X onto a closed subvariety V of some $\mathbf{A}^n$ and a rational representation of G on $\mathbf{k}^n$ preserving V.

Now that the connection between linear algebraic groups and matrices has been established, it makes sense to apply tools from linear algebra to study these groups. To this end we introduce a consequence of the fairly well-known Jordan canonical form, called the *Jordan decomposition*. We need two definitions first. Call a linear map a on a finite-dimensional $\mathbf{k}$-vector space V *semisimple* if V has a basis of a-eigenvectors, or equivalently the matrix of a is diagonal with respect to a basis of V. The matrix of a with respect to any basis is then also called semisimple. Call a *nilpotent* if $a^s = 0$ for some s and *unipotent* if $a - 1$ is nilpotent; again these notions extend to matrices. Let M_n denote the set of $n \times n$ matrices over $\mathbf{k}$.

Lemma 2.4.2, p. 38

If S is a family of pairwise commuting matrices in M_n there is $x \in M_n$ with xsx^{-1} upper triangular for each $s \in S$. If in addition the matrices in S are semisimple then we can choose $x \in GL_n$ with xsx^{-1} diagonal for every s; in particular, the sum or product of two commuting semisimple matrices is again semisimple.

Proof.

This is clear if S consists of scalar multiples of the identity. Otherwise there is $s \in S$ with an eigenspace W that is a proper subspace of $V = \mathbf{k}^n$; since the matrices of S commute pairwise, W is stabilized by all of them. Arguing by induction on dimension, the assertion holds for the actions of the matrices in S on both W and the quotient V/W, whence it also holds for V. The second assertion is proved similarly, writing V is a direct sum of eigenspaces for s. $\square$

One easily checks that the product of two commuting nilpotent matrices is again nilpotent, and likewise for unipotent matrices.

Then we have

Proposition 2.4.4, p. 32: additive Jordan decomposition

Let a be a linear transformation of $V = \mathbf{k}^n$.

- There are unique commuting transformations a_s, a_n with a_s semisimple, a_n nilpotent, and $a = a_s + a_n$.
- There are polynomials P, Q without constant term such that $a_s = P(a), a_n = Q(a)$.
- if $W \subset V$ is a-stable, then it is also stable under a_s, a_n and the restrictions of a_s, a_n to W give its Jordan decomposition. Similarly for quotients.
- If $\phi : V \to W$ and $b : W \to W$ are linear maps and if $\phi a = b\phi$, then $\phi a_s = b_s\phi, \phi a_n = b_n\phi$.

Proof.

Let $a_1, \ldots, a_m$ be the distinct eigenvalues of a and for each i let $V_i = \ker(a - a_i)^n$, the generalized a_i-eigenspace of a; then V_i is a-stable and V is the direct sum of the V_i. By the Chinese Remainder Theorem there is $P \in \mathbf{k}[T]$ with $P(T) \equiv 0 \mod T, P(T) \equiv a_i \mod (T - a_i)^n$; set $a_s = P(a)$. By the construction a_s acts by the scalar a_i on each V_i, so semisimply on V, and $a_n = a - a_s$ is nilpotent on each V_i, so on all of V. This proves the first two assertions, apart from the uniqueness. Now if $a = b_s + b_n$ is another Jordan decomposition of a, then b_s, b_n must commute with a_s, a_n, since the latter are polynomials in a, whence $a_s - b_s = a_n - b_n$ is simultaneously semisimple and nilpotent, so is 0. The third assertion follows form the construction. Finally the fourth assertion follows since ϕ factors as the injection $V \to V \oplus W$ given by $v \to (v, \phi(v))$ followed by the surjective projection $V \oplus W \to W$ and the third assertion proves the last one for both injections and surjections. $\square$

Corollary 2.4.5, p. 33: multiplicative Jordan decomposition

Given $a \in GL(V)$ there are unique commuting transformations $a_s, a_u \in GL(V)$ with a_s semisimple, a_u unipotent, and $a = a_s a_u$.

Indeed, if $a = a_s + a_n$ is the additive Jordan decomposition, then a_s must be invertible since a is and then, setting $a_u = 1 + a_s^{-1} a_n$, we have $a = a_s a_u$ with a_u unipotent; conversely, if $a = a_s a_u$ is the multiplicative Jordan decomposition, then $a = a_s + a_s(a_u - 1)$ is the additive one. It is also easy to check that the direct sum and tensor products of additive or multiplicative Jordan decompositions are again Jordan decompositions.

More generally, if V is a not necessarily finite-dimensional $\mathbf{k}$-vector space stabilized by a linear transformation a, then we say that a is *locally finite* if every $v \in V$ lies in a finite-dimensional a-stable subspace. In this case we call a semisimple if its restriction to any finite-dimensional a-stable subspace is semisimple. Similarly, a is *locally nilpotent* if its restriction to any finite-dimensional a-stable subspace is nilpotent; we define *locally unipotent* transformations similarly. Any locally finite transformation a on V admits a unique additive Jordan decomposition, and if invertible on V, a unique multiplicative decomposition, whose restriction to any finite-dimensional a-stable subspace is the relevant Jordan decomposition on that subspace; the above results show that any two such restrictions have well-defined decompositions agreeing wherever both are defined.