Lecture 12-1: The Isomorphism Theorem

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For the remainder of the course we will return to the text, sketching in very broad outline the main ideas in the proofs that the root datum of a reductive group determines the group up to isomorphism, and that every abstract root datum is the root datum of a reductive group.

Given the root datum $D = (X, R, \check{X}, \check{R})$ of a reductive group G relative to a maximal torus T, our first goal is to show that D determines G up to isomorphism. To this end we fix a realization $(u_\alpha)_{\alpha \in R}$ of the root system R in G (recall the definition on p. 133). Given $\alpha, \beta \in R$ we have we have the structure constants $c_{\alpha,\beta;i,j} \in \mathbf{k}$ defined for positive integers i, j by the equation $u_\alpha(x)u_\beta(y)u_\alpha(x)^{-1}u_\beta(x)^{-1} = \prod_{i,j:i\alpha+j\beta \in R} u_{i\alpha+j\beta}(c_{\alpha,\beta;i,j}x^i y^j)$ for all $x, y \in \mathbf{k}$ (p. 156); here the order of the factors is prescribed by a total order of R fixed in advance. (In particular, if $i\alpha + j\beta$ is never a root for any $i, j > 0$, then all $u_\alpha(x)$ commute with all $u_\beta(y)$.) If $(c'_{\alpha,\beta;i,j})$ is the set of structure constants arising from another realization $(u'_\alpha)_{\alpha \in R}$ then there are $c_\alpha \in \mathbf{k}^*$ such that $c_\alpha c_{-\alpha} = 1$ and $c'_{\alpha\beta;i,j} = c_\alpha^{-1} c_\beta^{-1} c_{i\alpha+_j\beta} c_{\alpha,\beta;i,j}$; we call the structure constants $(c_{\alpha,\beta;i,j})$ and $(c'_{\alpha,\beta;i,j})$ *equivalent* in this situation. For convenience we set $c_{\alpha,\beta;0,1} = c_{\alpha,\beta,1,0} = 1$ for all roots α, β.

The proof that D determines G is a long and intricate calculation. We will refer to the text for most of it, just stating the main results used along the way. Let $\alpha, \beta \in R$ be arbitrary. As in Lemma 8.1.4 (1), set $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$; this element normalizes T and represents the reflection s_α on R. For $\beta \in R, x \in \mathbf{k}$ define $u'_\beta(x) = n_\alpha u_{s_\alpha.\beta}(x)n_\alpha^{-1}$; then (u'_β) is another realization of R. Hence there is $d_{\alpha,\beta} \in \mathbf{k}^*$ such that $n_\alpha u_\beta(x)n_\alpha^{-1} = u_{s_\alpha.\beta}(d_{\alpha,\beta}x)$. Now if α, β are independent roots then a simple calculation using root systems of rank 2 shows that the set if integers i such that $\beta + i\alpha$ is a root is the intersection of an interval $[-c, b]$ with $\mathbb{Z}$, for some $c, b \geq 0$. We call the succession of roots $\beta - c\alpha, \ldots, \beta + b\alpha$ the *α-string through β* (p. 156). Then we have

Lemma 9.2.2, p. 156

- $d_{\alpha,\beta} = \sum_{i=\max(0,c-b)}^{c} (-1)^i c_{-\alpha,\beta;i,1} c_{\alpha,\beta-i\alpha;i+b-c,1}$;
- $d_{-\alpha,\beta} = (-1)^{\langle\beta,\check{\alpha}\rangle} d_{\alpha,\beta}$, $d_{\alpha,\beta} d_{-\alpha,-\beta} = d_{\alpha,\beta} d_{\alpha,s_\alpha.\beta} = (-1)^{\langle\beta,\check{\alpha}\rangle}$, $d_{\alpha,\beta} d_{\alpha,-\beta} = 1$, $d_{\alpha,\alpha} = -1$;
- $n_\alpha n_\beta n_\alpha^{-1} = (s_\alpha.\beta)\check{}(d_{\alpha,\beta}) n_{s_\alpha.\beta}$.

The proof of this result is given on pp. 157-8; it uses a rational representation of $SL_2(\mathbf{k})$. Next we have

Lemma 9.2.3, p. 158

Let $\alpha\beta, \gamma \in R$ and assume that β, γ are linearly independent.

- $c_{s_\alpha.\beta, s_\alpha.\gamma;i,j} - d^{-i}_{\alpha,\beta} d^{-j}_{\alpha,\gamma} d_{\alpha,i\beta+j\gamma} c_{\beta,\gamma;i,j}$ is uniquely determined by the $c_{\delta,\epsilon;i',j'}$ and the $d_{\alpha,\delta}$, where δ, ϵ are positive linear combinations of β, γ and $i' + j' < i + j$. In particular $c_{s_\alpha.\beta, s_\alpha.\gamma;1.1} = d^{-1}_{\alpha,\beta} d^{-1}_{\alpha,\gamma} d_{\alpha,\beta+\gamma} c_{\beta,\gamma;1,1}$.
- $c_{\gamma,\beta;i,j} = (-1)^i c_{\beta,\gamma;i,j}$ is uniquely determined by the $c_{\delta,\epsilon;i',j'}$ of the first part. In particular, $c_{\gamma,\beta;1,1} = -c_{\beta,\gamma;1,1}$.

This falls out of the calculation used to prove the preceding lemma.

Given linearly independent $\gamma, \delta \in R$ let R' be the intersection of the root system R and the two-dimensional subspace V' of the ambient vector space V spanned by γ and δ. Then R' is of rank two, so must be of type $A_1 \times A_1, B_2$ or G_2. If R' is of type $A_1 \times A_1$ then no combination $i\gamma + j\delta$ lies in R for any $i, j > 0$, whence there are no structure constants $c_{\gamma,\delta;i,j}$. Otherwise let α, β be a choice of simple roots for R'. The constants $c_{\gamma,\delta;i,j}$ are then determined by

Proposition 9.2.5, p. 158

- We can normalize the the realization (u_γ) such that all structure constants lie in $\mathbb{Z}$ (or in $\mathbb{Z}/p$, if the characteristic p of $\mathbf{k}$ is positive). We have $c_{\alpha,\beta;i,j} = 1$ for $i, j > 0$ and $i\alpha + j\beta \in R'$ and $d_{\alpha,\gamma}, d_{\beta,\gamma} = \pm 1$ for all positive roots $\gamma \in R'$ (relative to the choice α, β of simple roots).
- if R' is of type G_2 then $c_{\beta,3\alpha+\beta;1,1} = 1$
- The above properties uniquely determine all structure constants attached to pairs of roots in R'; moreover, we have $c_{\alpha,\beta;i,j} = c_{w.\alpha,w.\beta;i,j}$ for any w in the Weyl group W of R.

This is a lengthy but elementary calculation in each type, using the previous two lemmas (see pp. 159-161).

Next we address the elements $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ (for $\alpha \in R$) defined in Lemma 8.1.4; we have previously observed that this element represents the reflection s_α in W. Setting $t_\alpha = \check{\alpha}(-1)$, we have $n_\alpha^2 = t_\alpha$. Denote by $m(\alpha, \beta)$ the order of the product $s_\alpha s_\beta$ of reflections; recall that $m(\alpha, \beta) = 2, 3, 4$. or 6. A fairly elementary calculation shows that

Proposition 9.3.2, p. 162

If α and β are simple roots relative to a choice of positive roots, then the braid relation $n_\alpha n_\beta n_\alpha \ldots = n_\beta n_\alpha n_\beta \ldots$ holds, where there are $m(\alpha, \beta)$ factors on each side.

This is proved on pp. 162-3.

Now we are ready to give a presentation of G as an abstract group, given its root datum $(X, R, \check{X}, \check{R})$, realization $(u_\alpha)_{\alpha \in R}$, and structure constants $c_{\alpha,\beta;i,j}$; this is carried out on pp. 164-5. Fix a choice Δ of simple roots for R. We have seen that the structure constants may be normalized as in Lemma 9.2.5 and are then uniquely determined up to equivalence. It then turns out that $c_{\alpha,\beta;1,1} = \pm(c+1)$, where c is the largest nonnegative integer such that $\alpha - c\beta \in R$ (Proposition 9.5.3, p. 170). We begin by realizing a maximal torus of $\mathbf{G}$ as $\mathbf{T} = \hom(X, \mathbf{k}^*)$. For $\chi \in X$ we define a homomorphism $\overline{\chi} : T \to \mathbf{k}^*$ via $\overline{\chi}(\mathbf{t}) = \mathbf{t}(\chi)$. For $\lambda \in \check{X}$ define the homomorphism $\overline{\lambda} : \mathbf{k}^* \to \mathbf{T}$ via $\overline{\lambda}(x)(\chi) = x^{\langle \chi, \lambda \rangle}$ for $x \in \mathbf{k}, \chi \in X$. The Weyl group W (computable from the root datum, together with its actions on X and $\check{X}$) acts on $\mathbf{T}$ by $w.\mathbf{t}(\chi) = \mathbf{t}(w^{-1}.\chi)$. Now for $\alpha \in R, x \in \mathbf{k}$ we have a generator $\mathbf{u}_\alpha(x)$, corresponding to u_α; we impose the relations $\mathbf{u}_\alpha(x)\mathbf{u}_\alpha(y) = \mathbf{u}_\alpha(x+y)$.

Next we have the relations $\mathbf{u}_\gamma(x)\mathbf{u}_\delta(y)\mathbf{u}_\gamma(x)^{-1}\mathbf{u}_\delta(y)^{-1} = \prod_{i\gamma+j\delta\in R: i,j>0} \mathbf{u}_{i\gamma+j\delta}(c_{\gamma,\delta;i,j}x^i y^j)$ for $\gamma \neq \pm\delta$, using the given structure constants. It turns out to be enough to impose these relations only on roots γ, δ both lying in the span of two simple roots in Δ. Then we have the relations $\mathbf{t}\mathbf{u}_\gamma(x)\mathbf{t}^{-1} = \mathbf{u}_\gamma(\overline{\gamma}(\mathbf{t})x)$ for $\mathbf{t} \in \mathbf{T}, \gamma \in R, x \in \mathbf{k}$. Setting $\mathbf{n}_\gamma = \mathbf{u}_\gamma(1)\mathbf{u}_{-\gamma}(-1)\mathbf{u}_\gamma(1)$ we then require that $\mathbf{n}_\gamma\mathbf{u}_\gamma(x)\mathbf{n}_\gamma^{-1} = \mathbf{u}_{-\gamma}(-x), \mathbf{n}_\gamma^2 = \mathbf{t}_\gamma$, where $\mathbf{t}_\gamma(\chi) = (-1)^{\langle\chi,\check{\gamma}\rangle}$ for $\chi \in X$. We further require that $\mathbf{u}_\gamma(x)\mathbf{u}_{-\gamma}(-x^{-1})\mathbf{u}_\gamma(x) = \overline{\check{\gamma}}(x)\mathbf{n}\gamma$ and finally that $\mathbf{n}_\alpha\mathbf{n}_\beta\mathbf{n}_\alpha \ldots = \mathbf{n}_\beta\mathbf{n}_\alpha\mathbf{n}_\beta \ldots$, with $m(\alpha,\beta)$ factors on each side, if $\alpha, \beta \in R$.

Thus we define **G** as the group with generators **t** for **t** $\in$ **T** and all $\mathbf{u}_\alpha(x)$ as α runs over R and x over **k** and relations of the previous slide. Another long calculation, using the Bruhat decomposition, shows that $G \cong \mathbf{G}$ (pp. 165-7). In particular, any two reductive groups G, G' with isomorphic root data D, D' are isomorphic. Note however that this argument does *not* amount to a proof that a reductive group *exists* with any given root datum D, as it is not a priori clear that the abstract group **G** defined above has the structure of an affine variety.

We do however get some extra mileage out of this argument. First of all, since the structure constants $c_{\alpha,\beta;i,j}$ lie in $\mathbb{Z}$ (or in $\mathbb{Z}_p$ for some prime p) we see that an analogue of **G** exists over any basefield **k**, not necessarily algebraically closed. We will say more later about the groups that arise in this way; they are called *Chevalley groups*.

Next, the proof shows that *any automorphism* ϕ *of the torus* T *preserving* R *(thus also* $X, \check{X}$*, and* $\check{R}$*) induces an automorphism of the corresponding algebraic group* G*, unique up to conjugation by some* $t \in T$ (Theorem 9.6.2, p. 171). This is obvious for automorphisms of T coming from the action of W, since any such automorphism is realized by conjugation by a suitable element of the normalizer $N_G(T)$. But now some automorphisms of R do not arise in this way, namely those arising from nontrivial automorphisms of the Dynkin diagram D' corresponding to R. We will say more about such automorphisms, called *diagram automorphisms*, later; they play a crucial role in the proof that given any root datum D there is an algebraic group with that datum. We will also return to Chevalley groups on the last day of class.